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N. N. Kholshchevnikova
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On Concentrated Sets and N-Sets

N. N. KHOLSHCHEVNIKOVA*)

USSR

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We consider some known special subsets of the real line $\mathbb{R}$ and their use in decision of a few questions arised in the theory of trigonometrical series.

Let us remind some definitions.

A set $E \subset \mathbb{R}$ is called a Luzin set if it is uncountable and has countable intersection with every nowhere dense set.

A set $E \subset [0, 1)$ is called a N-set if there exists a trigonometric series in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi nx + b_n \sin 2\pi nx \tag{1}$$

which is absolutely convergent on E, but $\sum_{n=1}^{\infty} |a_n| + |b_n| = \infty$. Otherwise E is called an A.C.-set.

A set $E \subset [0, 1)$ is called a R-set if there exists a trigonometric series in the form (1) convergent on E, but $a_n^2 + b_n^2 \nrightarrow 0$.

It is known that N- and R-sets have measure zero and are meager or of first category. If a set is countable it is N- and R-set ([1], p. 173, 174, 736, 737, 757).

The question if any A.C.-set (not N-set) must have a cardinality continuum $\mathfrak{c}$ is unsolvable in ZFC (usual system of axioms of Zermelo-Fraenkel of set theory with axiom of choice).

It is trivial that assuming the continuum hypothesis CH the answer is positive. But assuming the failure of the CH the answer is ambiguous. Assuming Martin's axiom every set of cardinality less than $\mathfrak{c}$ is N- and R-set ([2]). (Z. Bukovska showed that it is truth by assuming Booth's lemma). On the other hand there is correctly

Theorem 1. *It is consistent that there exists a subset of cardinality less than* $\mathfrak{c}$ *of the Cantor set which is neither* N- *nor* R-*set.*

Proof. It is consistent with $\neg$ CH that there exists a Luzin set ([3], p. 205). Let us take a subset E of cardinality $\aleph_1$ (where $\aleph_1$ is the first uncountable cardinal) of a Luzin set on $[0, 1)$. A set E is a Luzin set on $[0, 1)$. Let us put their elements in

*) Veshnjakovskaya 19-255, Moskva, SSSR

order: $E = \{x_\alpha\}_{\alpha<\omega_1}$ where ω_1 is the first uncountable ordinal. As Steinhaus proved (see, for example [1], p. 739) any $t \in [0, 1]$ is representable in the form $t = x + y$, where x and y are from Cantor set K. Then any $x_\alpha \in E$ is $y_\alpha + z_\alpha$ where $y_\alpha, z_\alpha \in K$. Then $A = \{y_\alpha\}_{\alpha<\omega_1} \cup \{z_\alpha\}_{\alpha<\omega_1} \cup \{0\}$ is the required set.

Really, let A be a N-set. It is known ([1], p. 756) that there exists a trigonometric series in the form $\sum_{n=1}^{\infty} b_n \sin 2\pi nx$ absolutely convergent on E, but $\sum_{n=1}^{\infty} |b_n| = \infty$. Then for any $x_\alpha \in E$

$$\sum_{n=1}^{\infty} |b_n \sin 2\pi n x_\alpha| = \sum_{n=1}^{\infty} |b_n \sin 2\pi n(y_\alpha + z_\alpha)| \leqq$$

$$\leqq \sum_{n=1}^{\infty} |b_n \sin 2\pi n y_\alpha| + \sum_{n=1}^{\infty} |b_n \sin 2\pi n z_\alpha| < \infty .$$

But E can not be N-set as it is not meager. This contradiction proves that A is not N-set, i.e. A is an A.C.-set.

Let now A be a R-set. As $0 \in A$ then there exists a sequence $\{l_\nu\}$ of natural numbers such that $\lim_{\nu\to\infty} \sin l_\nu 2\pi x = 0$ for $x \in A$. But then $\lim_{\nu\to\infty} \sin 2\pi l_\nu x_\alpha = 0$ for $x_\alpha \in E$ (see [1], p. 741). It is impossible since E is not meager. So, A is not R-set and we are done.

Further we consider concentrated sets, defined by Besicovitch in 1934.

A set X is concentrated on a set A iff for any open set G if $A \subset G$, then $X \setminus G$ is countable.

On the whole there were studied sets concentrated on a countable set (see, for example [3], § 3).

Let us formulate some simple propositions on concentrated sets.

Proposition 1. *Let $A \subset \mathbb{R}$ and X be concentrated on A, then $X \setminus A$ is totally imperfect, i.e. doesn't contain any perfect set. If the complement of A is totally imperfect, then any set is concentrated on A.*

Proposition 2. [CH]. *Let $A \subset \mathbb{R}$, $A \,\bar{\in}\, G_\delta$. Then there exists a set X concentrated on A such that $X \setminus A$ is uncountable.*

We will need some definitions.

A set of reals X has universal measure zero iff for every atomless measure μ on the Borel sets there is a Borel set of μ-measure zero covering X.

A set of reals X is called universally measurable if for any given positive Borel measure μ it equals to a Borel set modulo a set of μ-measure zero.

Borel, analytic and coanalytic sets are universally measurable.

Proposition 3. *If a set $A \subset \mathbb{R}$ is universally measurable and X is concentrated on A, then $X \setminus A$ has universal measure zero.*

Proposition 4. *If a set $A \subset \mathbb{R}$ is analytic, but not Borel set, then there exists a set X concentrated on A such that cardinality of $X \setminus A$ is $\aleph_1$.*

We obtain such set selecting by one point from each of the constituents of a complement of A. Such a construction was used by Luzin to construct a perfectly meager set ([4], p. 283).

In [5] it is proved the next

Theorem 2. [MA + $\neg$ CH]. *Let* $A \subset \mathbb{R}$, $A \in F_\sigma$ *and* X *is concentrated on* A. *Then* $X \setminus A$ *is countable.*

Let us consider a question if a set concentrated on an N-set is itself an N-set. This question is unsolvable in ZFC.

Really, assuming CH there exists a Luzin set X on $[0, 1]$, which is concentrated on the set of rationals being a N-set, but X is not an N-set.

By theorem 2 it can be proved

Theorem 3. [MA + $\neg$ CH]. *A set concentrated on an* N-*set is an* N-*set.*

In [5] it was expressed the conjecture on independence of the statement: if $A \subset \mathbb{R}$ is Borel set and X is concentrated on A then $X \setminus A$ is countable.

V. Malychin and A. Roslanowski suggested the proof of the next theorem in which follows that this conjecture is true and even more strong statement is true.

Theorem 4. *If* $\mathfrak{b} > \aleph_1$ *then the following statement is true*:

If A *is an analytic set on* $\mathbb{R}$, $X \subset A$ *and* $|X \cap F| \leqq \aleph_0$ *for any compact subspace* $F \subset A$, *then* X *is countable.*

Here $\mathfrak{b}$ is the least cardinality of unbounded sets in ω^ω, where ω^ω is the space of all functions from ω to ω given the product topology. Remark, that $\mathfrak{b} > \aleph_1$ assuming MA + $\neg$ CH.

Proof of the Th. 4. Since A is an analytic set, then there exists a continuous map $f: \omega^\omega \to A, f(\omega^\omega) = A$. Let X be uncountable and Y is a subset of X, $|Y| = \aleph_1$. Let us put elements of Y in order: $Y = \{x_\alpha\}_{\alpha<\omega_1}$. Let us take $z_\alpha \in f^{-1}(x_\alpha)$. The set $\{z_\alpha\}_{\alpha<\omega_1}$ is bounded in ω^ω, since $\mathfrak{b} > \aleph_1$ and therefore is contained in a countable union of compact sets $\bigcup_{k=1}^{\infty} \Phi_k$ (see, for example [3], p. 215). Since f is continuous, then $f(\Phi_k)$ are also compact sets in A and hence Y is covered by countable numbers of compact sets and so Y is countable. This contradiction proves that X is countable.

From here it follows

Theorem 5. *Let* $\mathfrak{b} > \aleph_1$. *If* A *is a coanalytic set on* $\mathbb{R}$ *and* X *is concentrated on* A *then* $X \setminus A$ *is countable.*

Corollary. [$\mathfrak{b} > \aleph_1$]. *An analytic set* A *is a Borel set iff for every set* X *concentrated on* A *a set* $X \setminus A$ *is countable.*

Bibliography

[1] Bary N. K., Trigonometric series. Moscow 1961.

[2] Kholshchevnikova N. N., On uncountable R- and N-sets. Mat. Zametki 38, 2 (1985), 270—277.

[3] MILLER A. W., Handbook of Set-Theoretic topology. Chapter 5, Elsevier Science Pub. B. V., 1984.
[4] LUZIN N. N., Collection of works. V. 2, Moscow 1958.
[5] KHOLSHCHEVNIKOVA N. N., On some thin sets in the theory of functions and topology. Trudy Mat. Inst. AS USSR, to appear.