Miguel A. F. Sanjuán
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Chaos & Pseudochaos: Some Basic Remarks

M. A. F. SANJUÁN

Spain*)

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A basic introduction to the period doubling bifurcation route to chaos is given. The connection between numerical trajectories and true trajectories of a dynamical system, which present chaos, is discussed. The discretization procedure of continuous dynamical systems which model natural phenomena, and the corresponding properties which both systems, continuous and di crete, share, are presented. Finally some comments about the possible appearance of spurious in tabilities, pseudochaos, in the discrete model are sketched.

1. Period-Dubbing route to chaos

Dynamical systems attempt to understand processes in motion. Among the different processes we can consider, we have the motion of stars in galaxies, the ups and downs of the stockmarket, the weather forecasting, the chemical reactions, the ri e and fall of the populations, the mechanical oscillations, etc.

But the principal aim is to predict the state of the dynamical system at a further time. Thus, a simple question arises: Are dynamical systems predictable? The answer is also simple: Some yes, other no. But why is the question this way? Is it due to the very many variables included in the dynamical system? Well, this may be true, but it is not always the case, i.e., the answer is not complete. The very reason is chaos, which seems to appear in very simple dynamicl systems. With it, we mean that very simple systems depending on one variable, may behave unpredictably. There exists the hope of understanding more complex phenomena related to natural and applied sciences, with these simple dynamical systems.

In order to have a bit clearer idea of how chaos arises, we will consider the family of quadratic functions.

We have $Q_c(x) = x^2 + c$, where c is a parameter. This is one of the simplest nonlinear functions, and among the many different parameters we concentrate our-

*) Departamento de Física, E. T. S. de Arquitectura, Universidad Politécnica de Madrid, 28040 Madrid, Spain.

selves in those parameters with non-escaping orbits. For $-2 \leqq c \leqq 1/4$, there is an interval of values for which the orbits of $Q_c(x)$ do not escape. All this can be analytically explained. From $x^2 + c = x$, it can be easily proved that there exists two fixed points $p, q \in \Re$, when $c \leqq 1/4$:

$$p = \frac{1 + \sqrt{(1 - 4c)}}{2}, \quad q = \frac{1 - \sqrt{(1 - 4c)}}{2}$$

For $c = 1/4$ we have the so-called tangent bifurcation, which means that this first fixed point splits into two fixed points as c decreases.

For $-3/4 \leqq c \leqq 1/4$ all orbits tend toward an attracting fixed point q. For $c = -3/4$ we have the so-called period-doubling bifurcation. The attracting fixed point disappears and an attracting cycle of period 2 appears. For $c = -5/4$ the attracting cycle of period 2 becomes repelling, and a cycle of period 4 is born. And this continues while c decreases.

Thus at each stage a cycle of period 2^n becomes repelling, and an orbit of period 2^{n+1} is born. This is the so-called *period-doubling route to chaos*, which seems to appear in a great deal of dynamical systems.

For $c = -2$, then we have Q_{-2}, that has at least 2^n fixed points in the interval $-2 \leqq x \leqq 2$. However it has now *infinitely many periodic points*, and the orbits move about randomly. This is a chaotic quadratic function. [1]

In this way the orbit diagram is such, that the function has regions with period 2, 4, 8, 16. ... and a window of period 3, among other. When this window appears on a dynamical system, i.e., when the dynamical system has period three, it has been proved by Li & Yorke, in his famous paper. „Period three implies chaos" [2], that chaos exists. In fact it was here when for the first time the term chaos was introduced in the modern literature. Some years before the Russian mathematician Sharkovskii [3] had proved the same result.

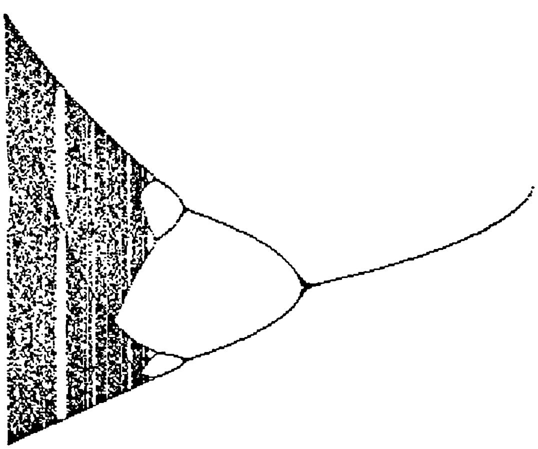

Fig. 1. Orbit Diagram of $Q_c(x)$ for $-2 \leqq c \leqq 1/4$

One very basic ingredient of chaos is the sensitive dependence on initial conditions, which means that no matter how close two orbits start out, after a few iterations, they will be very far apart. We will take the approach that a dynamical system is chaotic, when there exist Lyapunov exponents bigger than zero.

2. Shadowing Theorem

For a physical system with chaos, in what sense lies the computer-generated trajectory near the true trajectory of the system? In fact we must consider not only the necessary roundoff errors in computers, but the noise in experiments, errors in iterations and even the sensitive dependence on initial conditions, which is essential in chaotic orbits. A true orbit which stays near the pseudo-orbit is said to shadow the pseudo-orbit. In order to clarify the things we will use the following definitions [6], [7]:

Definition: $\{p_n\}_{n=a}^{b}$ is a δ_f – pseudotrajectory for f if $|p_{n+1} - f(p_n)| < \delta_f$, $a \leqq n \leqq b$, where δ_f is the noise amplitude.

Definition: $\{x_n\}_{n=a}^{b}$ is a true trajectory if $x_{n+1} = f(x_n)$, $a \leqq n \leqq b$

Shadowing theorem.
The true trajectory $\{x_n\}_{n=a}^{b}$ δ_x*-shadows* $\{p_n\}_{n=a}^{b}$ *on* $a \leqq n \leqq b$, *if* $|x_n - p_n| < \delta_x$ *for* $a \leqq n \leqq b$.

Definition; The pseudotrajectory $\{p_n\}_{n=a}^{b}$ has a glitch at iterate $n = N < b$, if for some relevant δ_x there exists a true trajectory that δ_x-shadows $\{p_n\}_{n=a}^{b}$ for $0 \leqq n \leqq N$, but no true trajectory that δ_x-shadows it for $0 \leqq n \leqq N_1$, when $N_1 > N$.

The shadowing theorem was originally proved by Anosov and Bowen [4], [5], for hyperbolic maps. Their proof is of no practical use for computer experiments due to the order of magnitudes of the δ's. On the other hand most physical dynamical systems are not hyperbolic.

Very recently it has been proved by Yorke and coworkers [6], [7] for non-hyperbolic chaotic processes, which are the typical systems found in nonlinear dynamics. The very reason why shadowing works here, lies in the hyperbolicity along the pseudotrajectory. Even they formulate a conjecture to indicate the relative magnitudes of the quantities involved in the dynamical system: For a typical 2D Hamiltonian map with a chaotic trajectory and a small $\delta_f > 0$, it is expected a $\delta_x \leqq \leqq \sqrt{\delta_f}$ for a trajectory of length $N \approx 1/\sqrt{\delta_f}$.

3. Discretization of a continuous dynamical system

Natural phenomena are usually modelled through the help of nonlinear ordinary differential equations, which have no exact solutions. It is necessary then to build a discrete model for computational purposes. One question so arises: Do the discrete system share the same properties of the original continuous system?

In principle numerical models represent the underlying continuous dynamical system when the step size tends to zero. The step size should be small enough, but, how small enough?

It is known that for some numerical models with large enough step sizes, a type of spurious instability, computational chaos or pseudochaos is produced, which has no relation to the dynamics of the continuous model.

We will consider here two simple examples. The first one is the pendulum which is a continuous system and which has been for long used as a paradigm of Hamiltonian nonlinear systems, and with the help of a simple algorithm we build a discrete system from it, with quite different properties.

Pendulum	*Standard Mapping*
$\dot{x} = y$	$I_{n+1} = I_n - \mathrm{K} \sin \theta_n$
$\dot{y} = -\omega_0^2 \sin x$	$\mathrm{K} \ll 1$
$x + \omega_0^2 \sin x = 0$	$\theta_{n+1} = \theta_n + I_{n+1}$

In order to obtain the so-called standard mapping, which is also a paradigm for Hamiltonian discrete nonlinear systems; we have just considered the following algorithm:

$$\dot{x} = \frac{x_{n+1} - x_n}{\Delta t} \qquad \ddot{x} = \frac{\mathrm{d}p}{\mathrm{d}t}$$

where $\Delta t \ll$ small and with $\mathrm{K} = \omega_0^2(\Delta t)^2 \ll 1$. We have used the following change of variables $\Delta t p \to \mathrm{I}$, $x \to \theta$.

But now we note something surprising. While the continuous case is completely integrable, i.e. with exact solutions and with invariant curves in phase space, the discrete case has very different properties.

For the standard mapping we have the following pattern of behaviour depending on the value of the constant K [8]:

1. If $\mathrm{K} = 0$, the system is completely integrable.
2. If $\mathrm{K} \neq 0$, but $\mathrm{K} \to 0$, then K.A.M. curves still are preserved and stochastic layers exist in phase space.
3. If $\mathrm{K} \to \infty$, then the system is fully chaotic.

This clearly shows that in the process of discretization we obtain a system whose dynamical properties are far apart from the one which possesses the continuous system.

We consider now the Rössler model [9], which is a system that comes from the chemical dynamics. We have a system of three ordinary differential equations.

$$\begin{aligned}
\dot{x} &= -(y + z) \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}$$

where a, b and c are constants. After using the backward Euler algorithm $dw/dt = f(w)$, we obtain the discrete dynamical system $w_{n+1} - hf(w_{n+1}) = w_n$, where h can act as a control parameter and for $h \to 0$ in principle we go back to the continuous system.

If for h small the trajectories are chaotic, some questions can be formulated:

1. Is the original continuous dynamical system also chaotic?
2. Does the computed trajectory corresponds to the true trajectory?

The first question can be answered affirmatively only in the case of arbitrarily small step sizes. For the last question shadowing applies, but still even assuming shadowing, at least for small h, there is no guarantee that the chaotic attractor present in the discrete system is continuous in h.

In the present case the discretization has precisely the same fixed points as the continuous dynamical system, which is a property not shared by all numerical models.

If we vary the step size h as a control parameter it results that: There exists a value h^*, which is a Hopf bifurcation of the discrete system and such that:

For $h > h^*$ appears a fixed point which is stable.

For $h = h^*$ the Lyapunov exponent is equal to zero.

For $h < h^*$ appears a fixed point which is unstable.

So as a conclusion we can affirm that:

I. Numerical methods can generate numerical instabilities, pseudochaos, which are not present in the underlying continuous dynamical system.

II. Some numerical methods can also suppress chaos via spurious stability, by simply increasing the step size of the integrator method.

III. Pseudochaos can be generated also by round-off errors, just as a small effect contributing to the calculation of the Lyapunov exponents.

As a consequence of all this we can say that it is necessary to pay more attention to the discretization method, and realize that different algorithms give rise to different discrete models, which have or not the same fixed points. Different step sizes give also rise to different behaviours. The identification of the sources of cancellation errors along with the precision of computational calculations results is also necessary.

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