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Poincaré and Domain Invariance Theorem

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An elementary proof is given of Brouwer's theorem on the invariance of a domain. It is shown that this theorem is an easy consequence of the Bolzano-Poincaré intermediate value theorem.

The purely topological problem of the invariance of a domain was arisen from the geometrical theory of analytic functions. It was solved by Brouwer [3] in 1911. In paper [4] which is a record of Brouwer's lecture delivered 27 September 1911, at a meeting in Karlsruhe of the German Mathematical Society Brouwer wrote "... Poincaré gives a proof of the existence of a linearly-polymorphic function on a Riemann surface by the method of continuity, accepting without discussion the following two assertions:

Theorem 1. *Classes of a Riemann surface of genus g form a $(6g - 6)$-dimensional manifold without singularities.*

Theorem 2. *A one-to-one and continuous image of an n-dimensional domain in an n-dimensional manifold again forms a domain.*

Due to a small change in the method we can avoid applying Theorem 1 ... and, thus it all reduces to a justification of Theorem 2 — the theorem of the invariance of domain, a proof of which I shall publish in the near future."

Theorem 2 usually is presented in an equivalent form as

Domain Invariance Theorem. *If $h : U \to R^n$ is a continuous one-to-one map from an open set $U \subset R^n$ then $h(U)$ is an open subset of R^n, too.*

We shall derive this theorem from the

Lemma. *Let $f : X \to R^n \setminus \{0\}$ be a continuous map from a compact subset $X \subset R^n$. Then for each $\varepsilon > 0$ and for each compact boundary subset $Y \subset R^n$ there exists a continuous map $F : X \cup Y \to R^n \setminus \{0\}$ such that $\|F(x) - f(x)\| < \varepsilon$ for each $x \in X$,*

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and the

Poincaré-Miranda Theorem. *Let $f: I^n \to R^n$, $f = (f_1, \dots, f_n)$, be a continuous map such that for each $i \leq n$, $f_i(I_i^-) \subset (-\infty, 0]$ and $f_i(I_i^+) \subset [0, +\infty)$. Then there exists a point $c \in I^n$ such that $f(c) = \mathbf{0}$,*

where $\mathbf{0} := (0, \dots, 0) \in R^n$, $I^n = [-a, a]^n$ is an *n-dimensional cube* and $I_i^- := \{x \in I^n : x_i = -a\}$ and $I_i^+ := \{x \in I^n : x_i = a\}$ its i-th *opposite faces.*

Let us leave until later the task of proving Lemma. A proof of the Poincaré-Miranda theorem and its applications one can find in [6]. For convenience of the reader we shall repeat the proof. In fact, this theorem is due to Poincaré [8] who in 1883 announced without proof the following result (in Browder's translation [5]):

"Let $f_1, \dots, f_n$ be n continuous functions of n variables $x_1, \dots, x_n$: the variable x_i is subjected to vary between the limits $+a_i$ and $-a_i$. Let us suppose that for $x_i = a_i$, f_i is constantly positive, and that for $x_i = -a_i$, f_i is constantly negative; I say there will exist, a system of values of x for which all the f's vanish."

This theorem is sufficient to prove Brouwer's theorem. Theorem of Poincaré was rediscovered by Miranda [7], who in 1940 showed that it was equivalent to Brouwer's fixed point theorem. Now let us proceed to the

Proof of Domain Invariance Theorem. Fix $u \in U$. Without loss of generality we may assume that $u = \mathbf{0}$. Let $I^n := [-a, a]^n$ be an n-dimensional cube such that $I^n \subset U$. In order to prove the theorem it suffices to check that $b := h(\mathbf{0}) \in \operatorname{Int} h(I^n)$.

Since I^n is a compact space, the map $h|I^n$ is a homeomorphism from I^n onto $h(I^n)$. Therefore there exists $\delta > 0$ such that $h^{-1}(B(b, 2\delta)) \subset \operatorname{Int} I^n$. Suppose that $b \in \partial h(I^n)$. Then one can choose a point $c \in B(b, \delta) \setminus h(I^n)$. It is clear that b belongs to the ball $B(c, \delta)$ and $h^{-1}(B(c, \delta)) \subset \operatorname{Int} I^n$. Let us put $X := h(I^n) \setminus B(c, \delta)$ and $Y := \partial B(c, \delta)$. Define a continuous map $l: h(I^n) \cup B(c, \delta) \setminus \{c\} \to X \cup Y$ such that

$$l(x) := \begin{cases} x & \text{if } x \in X, \\ c + \dfrac{x - c}{\|x - c\|} \cdot \delta & \text{if } x \in B(c, \delta) \setminus \{c\} \end{cases}$$

($l: B(c, \delta) \setminus \{c\} \to \partial B(c, \delta)$ is a retraction, see Figure 1).

Applying Lemma to $\varepsilon = a$ and to the map $h^{-1}|X: X \to r^n \setminus \{\mathbf{0}\}$ we can find a continuous map $g: X \cup Y \to R^n \setminus \{\mathbf{0}\}$ such that $\|g(x) - h^{-1}(x)\| < a$ for each $x \in X$.

Finally, define $f = (f_1, \dots, f_n): I^n \to R^n \setminus \{\mathbf{0}\}$ as $f := g \circ l \circ h$. This map does not assume the value $\mathbf{0}$. On the other hand it satisfies the assumptions of Poincaré's theorem; $f_i(I_i^-) \subset (-\infty, 0)$ and $f(I_i^+) \subset (0, +\infty)$. To see this, fix $t \in I_i^-$. Note that $l(h(t)) = h(t)$, because $h(t) \in X$. Let us check; $\|f(t) - t\| = \|g(l(h(t))) - h^{-1}(h(t))\| = \|g(h(t)) - h^{-1}(h(t))\| < a$. Since $t_i = -a$ we get $|f_i(t) - t_i| = |f_i(t) - (-a)| \leq \|f(t) - t\| < a$. This implies that $f_i(t) < 0$. Similarly one can

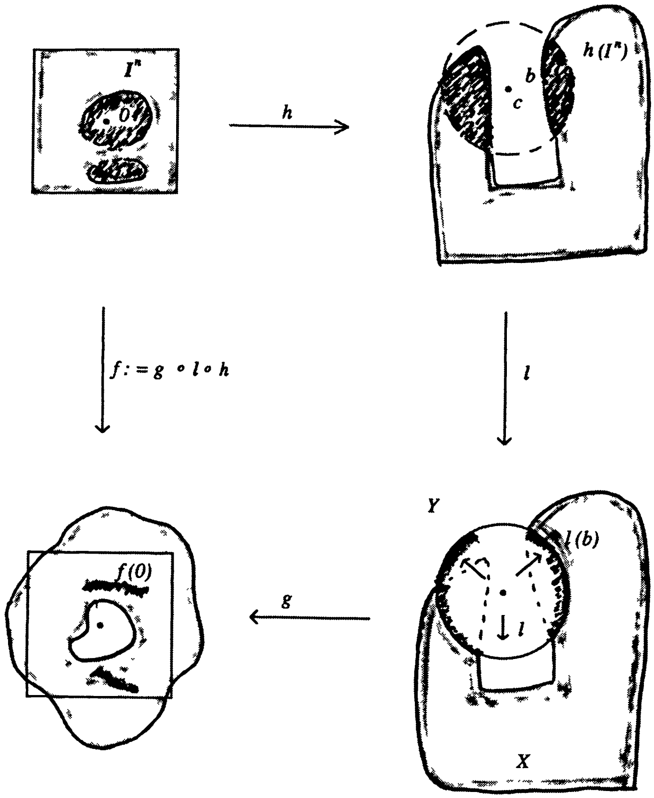

Fig. 1

check the other case. Thus, according to the Poincaré theorem f must assume $\mathbf{0}$, a contradiction. ■

Remark. For the reader who prefers analytical methods Lemma is superflous. Since $Y := \partial B(c, \delta)$ is a compact set of measure zero, it suffices to know that maps of class C^1 preserve measure zero of compact sets. Now, applying the Weierstrass Approximation Theorem it is easy to find a polynomial map $g : R^n \to R^n$ such that $\mathbf{0} \notin g(X \cup Y)$ and $\|g(x) - h^{-1}(x)\| < a$ for each $x \in X$.

To see this, let us choose η; $0 < 2\eta < a$, such that $h^{-1}(X) \cap B(\mathbf{0}, 2\eta) = \emptyset$ and let $p : R^n \to R^n$ be a polynomial map such that $\|p(x) - h^{-1}(x)\| < \eta$ for each $x \in X$. Since $p(Y)$ is a set of measure zero there is a point $d \in B(\mathbf{0}, \eta) \setminus p(Y)$. Define $g : X \cup Y \to R^n$;

$$g(z) := p(z) - d \quad \text{for each } z \in X \cup Y.$$

Note that $\mathbf{0} \notin g(Y)$ if and only if $d \notin p(Y)$. Let us see that for each $x \in X$: $|h^{-1}(x) - g(x)\| \leq | h^{-1}(x) - p(x)\| + \|d\| < 2\eta$. Since $h^{-1}(X) \cap B(\mathbf{0}, 2\eta) = \emptyset$ we infer that $\mathbf{0} \notin g(X)$. Now, it is clear that $\mathbf{0} \notin g(X \cup Y)$. ■

The Poincaré-Miranda Theorem. Fix $n, k = 1, 2, \ldots$ and $a > 0$. Using the Certesian notation let $e_i := (0, \ldots, 0, a/k, 0, \ldots, 0)$, $e_i(i) = a/k$, be the i-th basic vector. Let $Z_k := \{j \cdot \frac{a}{k} : j \in Z\}$, where Z is the set of integers. Denote Z_k^n to be the Cartesian product of n-copies of the set Z_k;

$$Z_k^n := \{z : \{1, \ldots, n\} \to Z_k \mid z \text{ is a map}\}$$

Let $P(n)$ be the set of permutations of the set $\{1, \ldots, n\}$.

Definition. *An ordered set $S = [z_o, \ldots, z_n] \subset Z_k^n$ is said to be a (n-dimensional) simplex if there exists a permutation $\alpha \in P(n)$ such that*

$$z_1 = z_0 + e_{\alpha(1)},\ z_2 = z_1 + e_{\alpha(2)},\ \ldots,\ z_n = z_{n-1} + e_{\alpha(n)}\,.$$

Observation. *Let $S = [z_0, \ldots, z_n] \subset Z_k^n$ be a simplex. Then for each point $z_i \in S$ there exists exactly one simplex $T = S[i]$ such that*

$$S \cap T = \{z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}\,.$$

Proof. We shall define the i-neighbour $S[i]$ of the simplex S (see Figure 2) as

1) If $0 < i < n$, then $S[i] := [z_0, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n]$, where $x_i = z_{i-1} + (z_{i+1} - z_i) = x_{i-1} + e_{\alpha(i+1)}$.
2) If $i = 0$, then $S[0] := [z_1, \ldots, z_n, x_0]$, where $x_0 = z_n + (z_1 - z_0)$,
3) If $i = n$, then $S[n] := [x_n, z_0, \ldots, z_{n-1}]$, where $x_n = z_0 - (z_{n+1} - z_n)$,

We leave to the reader the prove that the simplexes $S[i]$ are well defined and that they are the only possible i-neighbours of the simplex S. ■

Any subset $[z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n] \subset S$, $i = 0, \ldots, n$, is said to be ($(n-1)$-dimensional) i-face of the simplex S. A subset $C \subset Z_k^n$ of the form

$$C := C(k) = \left\{x \in I^n : x_j = j \cdot \frac{a}{k},\ \text{where } j = 0, \pm 1, \ldots, \pm k\right\}$$

is said to be a *combinational n-cube.*

Define the i-th opposite faces of C;

$$C_i^- := \{z \in C : z(i) = -a\}, \quad C_i^+ := \{z \in C : z(i) = a\}$$

and the boundary

$$\partial C := \bigcup \{C_i^- \cup C_i^+ : i = 1, \ldots, n\}$$

From the above observation we get the following

Observation. *Any face of a simplex contained in the cube C is a face of exactly one or two simplexes from C, depending on whether or not it lies on the boundary ∂C.*

Proof the Poincaré-Miranda Theorem. For each $i = 1, \ldots, n$ define $H_i^- := f_i^{-1}(-\infty, 0]$, $H_i^+ := f_i^{-1}[0, \infty)$. Since for each sequence of simplex $S_k \subset C(k)$, *diameter* $S_k \to 0$ as $k \to \infty$, in order to prove the theorem it suffices to show that for each k there exists a simplex $S_k \subset C(k)$ such that

$$(1)\qquad H_i^- \cap S_k \neq \emptyset \neq H_i^+ \cap S_k \quad \text{for each } i = 1, \dots, n\,.$$

Indeed, using the compactness argument we infer that the intersection

$$H := \bigcap \{H_i^- \cap H_i^+ : i = 1, \dots, n\}$$

is not empty set. It is clear that $f(c) = \mathbf{0}$ for each $c \in H$.

Define a map $\varphi : I^n \to \{0, \dots, n\}$;

$$(2)\qquad \varphi(x) := \max\{j : x \in \bigcap_{i=0}^{j} F_i^+\},$$

where $F_0^+ := I^n$ and $F_i^+ := H_i^+ \setminus I_i^-$ for each $i = 1, \dots, n$. The map φ has the following properties:

$$(3)\qquad \text{if } x \in I_i^-, \text{ then } \varphi(x) < i, \text{ and if } x \in I_i^+, \text{ then } \varphi(x) \neq i - 1\,.$$

From (3) it follows that for each subset $S \subset I^n$;

$$(4)\qquad \varphi(S \cap I_i^\varepsilon) = \{0, \dots, n-1\} \text{ implies that } i = n \text{ and } \varepsilon = -.$$

Observe that from (2) and the fact that $I^n = H_i^- \cup H_i^+$, imply that

$$(5)\qquad \text{if } \varphi(x) = i - 1 \text{ and } \varphi(y) = i, \text{ then } x \in H_i^- \text{ and } y \in H_i^+\,.$$

Let us call a finite subset S of $l + 1$ points in the combinatorial cube $C = C(k)$ to be *proper* if $\varphi(S) = \{0, \dots, l\}$. From (1) and (5) it follows that the theorem will be proved if we show that for each k there exists a proper simplex $S \subset C(k)$. It will be proved that for each k the number ϱ of proper simplexes will be odd.

Our proof will be by induction on the dimensionality n of C. This is obvious for $n = 0$, because $C = \{0\}, \varphi(0) = 0$, $\varrho = 1$.

According to (4) any proper face $s \subset \partial C$ lies in C_n^- and by our induction hypothesis the number α of such faces is odd. Let $\alpha(S)$ denote the number of proper faces of a simplex $S \subset C$.

Now, if S is a proper simplex, clearly $\alpha(S) = 1$; while if S is not a proper simplex, we have $\alpha(S) = 2$ or $\alpha(S) = 0$ according as $\varphi(S) = \{0, \dots, n-1\}$ or $\{0, \dots, n-1\} \setminus \varphi(S) \neq \emptyset$.

Hence

$$(6)\qquad \varrho = \sum \alpha(S), \bmod 2$$

On the other hand, a proper face is counted exactly once or twice in $\sum \alpha(S)$ according as it is in the boundary of C or not.

Accordingly

$$(7)\qquad \sum \alpha(S) = \alpha, \bmod 2$$

hence

$$(8)\qquad \alpha = \varrho, \bmod 2\,.$$

But α is odd. Thus ϱ is odd, too. ■

We shall describe briefly, without proofs, one of simplicial methods which were initiated in topology by Poincaré.

A subset $S = \langle z_0, \dots, z_n \rangle \subset I^n$ is said to be an *n-dimensional geometrical simplex* (a *simplex* for brevity) if $z_0, \dots, z_n \in C$ and there exists a permutation $\alpha \in P(n)$ such that

(1) $z_1 = z_0 + e_{\alpha(1)}$, $z_2 = z_1 + e_{\alpha(2)}, \dots,$ $z_n = z_{n-1} + e_{\alpha(n)}$, and

(2) $S = \{x : x = \sum_{i=1}^{n} t_i \cdot z_i, \ \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0\}$.

The sum $\sum_{i=0}^{n} t_i z_i$, where $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0$, is said to be a *convex combination of points* $z_0, \dots, z_n$.

A family $Q = Q(k)$ consisting of the all n-dimensional geometrical simplexes is said to be a *regular triangulation* of the cube I^n (see Figure 2).

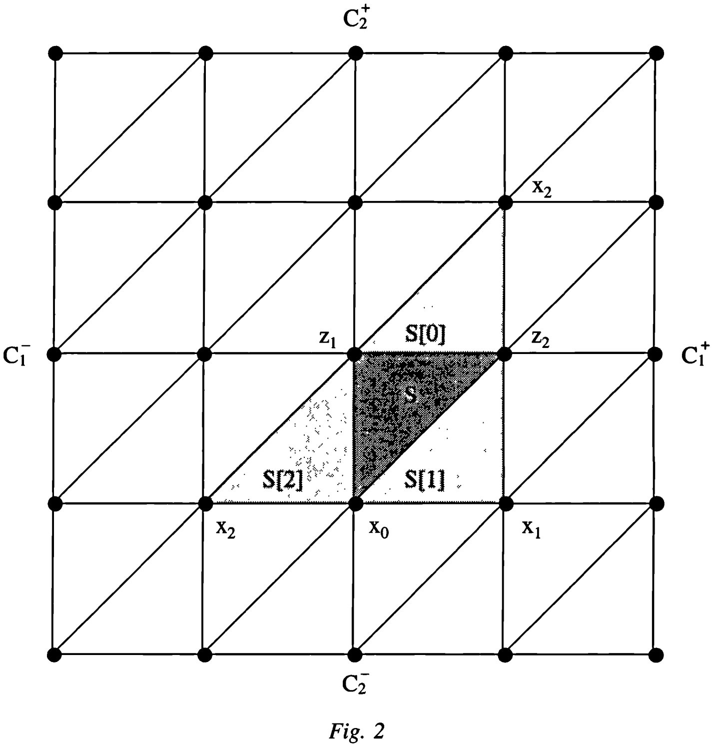

Fig. 2

In the proof of Lemma we shall use the following facts;

(a) For each point $x \in I^n$ there is a simplex $S \in Q$ such that $x \in S$. Each point $x \in \langle z_0, \dots, z_n \rangle$ is uniquely determined by its *barycentric coordinates* $t_i = t_i(x)$;

$$x = \sum_{i=1}^{n} t_i \cdot z_i, \quad \sum_{i=1}^{n} t_i = 1, \ t_i \geq 0,$$

which are continuous functions of x, $t_i : S \to [0, 1]$.

(b) Each map $h: C \to R^n$ uniquely determines a piece-wise (affine) linear continuous map $h: I^n \to R^n$;

$$h(x) = \sum_{i=0}^{n} t_i \cdot h(z_i), \quad x \in \langle z_0, \ldots, z_n \rangle,$$

where x is a convex combination of the points z_i, $x = \sum_{i=0}^{n} t_i \cdot z_i$.

Now, let us start the

Proof of Lemma. Fix $\varepsilon > 0$. Since $X \cup Y$ is a bounded subset of R^n there is an $a > 0$ such that $X \cup Y \subset I^n$, where $I = [-a, a]$. Extend the map f to a continuous map $g: I^n \to R^n$ and let us choose a δ, $0 < 2\delta < \varepsilon$, such that $f(X) \cap B(\mathbf{0}, 2\delta) = \emptyset$. Let P be a covering of R^n consisting of open balls of diameter less than δ. Since g is a uniformly continuous map there exists a regular triangulation $Q = Q(k)$ of I^n such that for each simplex $S \in Q$, $g(S)$ is contained in some ball $B \in P$. Define a piece-wise (affine) linear map $h: I^n \to R^n$ induced by the map $g|C$:

$$h(x) := \sum_{i=0}^{n} t_i g(z_i),$$

where $x \in \langle z_0, \ldots, z_n \rangle \in Q$ and x is a convex combination of the points z_i.

Observe that $\|f(x) - h(x)\| < \delta$, because for each $S \in Q$ there exists a ball $B \in P$ (being a convex set) such that $g(S) \subset B$. Since $h(S)$ is a convex set, in view of the definition of h we get that $h(S) \subset B$, too.

Now let us note that if the points $g(z_0), \ldots, g(z_n)$ are linearly dependent (in the affine sense) then the $h(S)$ lies in an $(n-1)$-dimensional hyperplane and therefore $h(S)$ is a compact boundary subset of R^n. If the points $g(z_0), \ldots, g(z_n)$ are linearly independent then $h(S \cap Y)$ is a compact boundary subset as a image of a compact boundary set under a linear homeomorphism from R^n onto R^n!

This yields that $h(Y)$ is a boundary set as a finite union of compact boundary sets $h(S \cap Y)$, $S \in Q$. From $f(X) \cap B(\mathbf{0}, 2\delta) = \emptyset$ and $\|f(x) - h(x)\| < \delta$ for each $x \in X$, we obtain that $h(X) \cap B(\mathbf{0}, \delta) = \emptyset$. Now, it is clear that we can choose a point $d \in B(\mathbf{0}, \delta) \setminus h(X \cup Y)$.

Define the map $F: X \cup Y \to R^n$ as $F(z) := h(z) - d$. Let us note that $\|F(x) - f(x)\| \leq \|h(x) - f(x)\| + \|d\| < 2\delta < \varepsilon$ for each $x \in X$ and $F(z) \neq \mathbf{0}$ for each $z \in X \cup Y$ (because $F(z) = \mathbf{0}$ implies $h(z) = d$, a contradiction with $d \notin h(X \cup Y)$). This completes the proof. ■

Domain Invariance Theorem implies

Dimension Invariance Theorem. *There is no continuous one-to-one map $f: R^n \to R^m$ for $m < n$.*

Proof. Define $h := i \circ f$, where $i: R^m \to R^n$, $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$ is an embedding. The set $h(R^n)$ is a boundary subset of R^n. On the other hand, according to Domain Invariance Theorem, it is an open subset of R^n, a contradiction. ■

The first correct proof that Euclidean spaces R^n and R^m are not homeomorphic unless n equals m was given in 1911 by Brouwer [2]. This theorem was very important in view of results of Cantor 1877 on existence of $1-1$ maps between R^n and R^m and a result by Peano 1890 implying existence of continuous maps from R^n onto R^m for $n < m$. In years 1911 – 1924 Lebesgue published not quite correct proofs of theorems on the invariance of dimensions and domains. This caused a quarrel between Brouwer and Lebesgue on the priority of results and involved some known mathematicians to public reactions. In 1924 Lebesgue gave Brouwer full credit for the invariance of dimension but claimed for himself the theorem of the invariance of domains. Lebesgue's papers, not quite correct, were not fruitless. They led to discovery of a notion of covering dimension.

Remarks. The Poincaré-Miranda theorem can be expressed also for non-continuous maps.

Assume that $f : X \to R^n$ is a map from a metric space X. The least number $\eta = \eta(f)$, $0 \leq \eta \leq \infty$, such that $\limsup \|f(x_m) - f(x)\| \leq \eta$ for each sequence $x_m \to x$, is said to be a number of discontinuity of the map f. It is clear that $\eta(f) = 0$ whenever f is continuous.

The following version of the Poincaré-Miranda theorem for not necessarily continuous maps holds:

Poincaré-Miranda Theorem. *Let $f : I^n \to R^n$, $f = (f_1, \dots, f_n)$, be a map such that for each $i \leq n$, $f_i(I_i^-) \subset (-\infty, 0]$ and $f_i(I_i^+) \subset [0, +\infty)$. Then there exists $c \in I^n$ such that for each $i \leq n$, $|f_i(c)| \leq \eta(f_i)$.*

Proof. For each $i = 1, \dots, n$ and $x \in I^n$ define

$$g_i(x) := d(x, f_i^{-1}(-\infty, 0]) - d(x, f_i^{-1}[0, +\infty)),$$

where $d(x, A) := \inf\{\|x - a\| : a \in A\}$ is the distance function from the set A.

The map $g : I^n \to R^n$, $g = (g_1, \dots, g_n)$, satisfies the assumptions of the Poincaré-Miranda theorem and therefore there exists $c \in I^n$ such that $g(c) = \mathbf{0}$.

This means that for each $i \leq n$,

$$d(c, f_i^{-1}(-\infty, 0]) = 0 = d(c, f_i^{-1}[0, +\infty)).$$

Fix $i \leq n$, and choose sequences of points $x_m, y_m \in I^n$; $x_m \to c$ and $y_m \to c$ such that $f_i(x_m) \leq 0$ and $f_i(y_m) \geq 0$ for each m. According to definition of the number of discontinuity we get that $|f_i(c)| \leq \eta(f_i)$. ■

The Bohl-Brouwer Fixed Point Theorem. *For any continuous map $g : I^n \to I^n$ there exists a point $c \in I^n$ such that $|c_i - g_i(c)| \leq \eta(g_i)$ for each $i \leq n$.*

Proof. Let us put $f(x) := x - g(x)$. The map f satisfies the assumptions of the Poincaré-Miranda Theorem and therefore there is a point $c \in I^n$ such that $|f_i(c)| \leq \eta(f_i) = \eta(g_i)$ for each $i \leq n$. ■

When children amused themselves by blowing bubbles with a soap solution and when the arising bubble burst then they first learn by experience the following consequence of the Poincaré-Miranda theorem.

Exploding Point Theorem. *Let $X \subset R^n$ be a compact subset such that $(-\delta, \delta)^n \subset X$ for some $\delta > 0$. Then each map $f : X \to X \setminus (-\delta, \delta)^n$ which is the identity map on the boundary of X, has an exploding point i.e., there are $c \in X$ and $j \leq n$ such that for each $\varepsilon > 0$ there are two points $x, y \in B(c, \varepsilon)$ with $f_i(x) \leq -\delta$ and $f_j(y) \geq \delta$.*

Proof. Let I^n be an dimensional cube such that $X \subset I^n$ and extend the map f to the map $f : I^n \to I^n$ such that $f(x) = x$ for each $x \in I^n \setminus X$. Similarly as in the preceding proof define

$$g_i(x) := d(x, f_i^{-1}(-\infty, -\delta]) - d(x, f_i^{-1}[\delta, +\infty)).$$

The map $g : I^n \to R^n$, $g = (g_1, \ldots, g_n)$ satisfies the assumption of the Poincaré-Miranda theorem and therefore there is $c \in I^n$ such that $g(c) = \mathbf{0}$.

Since $f(c) \notin (-\delta, \delta)^n$ there exists $j \leq n$ such that $|f_j(c)| \geq \delta$. This yields, $d(c, f_j^{-1}(-\infty, -\delta]) = 0$ or $d(c, f_j^{-1}[\delta, +\infty)) = 0$.

From $g_j(c) = 0$ we infer that $0 = d(c, f_j^{-1}(-\infty, -\delta]) = d(c, f_j^{-1}[\delta, +\infty))$. This implies that for each $\varepsilon > 0$ there exist points $x, y \in B(c, \varepsilon) \cap X$ such that $f_j(x) \leq -\delta$ and $f_j(y) \geq \delta$. ■

The effect of bursting bubbles as an illustration of Exploding Point Theorem can be observed also while a yeast dough is waiting to be ready for baking.

Exploding Point Theorem implies the Borsuk non-retraction theorem stating that there is no continuous map from a ball onto its boundary which keeps each point in the boundary fixed.

Conclusion. It is easily to observe that the Poincaré theorem can be strengthened to a weak version of the invariance of domains theorem:

If $f = (f_1, \ldots, f_n) : I^n \to R^n$ is a continuous map such that for each $i \leq n$; $f_i(I_i^-) \subset (-\infty, 0)$ and $f_i(I_i^+) \subset (0, +\infty)$, then $\mathbf{0} \in \operatorname{Int} f(I^n)$.

To prove this, note that by compatness of I^n and from the assumptions it follows that there exists $\delta > 0$ such that $f_i(I_i^-) \subset (-\infty, -\delta)$ and $f_i(I_i^+) \subset (\delta, +\infty)$ for each $i \leq n$. Now observe that for each $b \in J^n := [-\delta, \delta]^n$ the map $f_b(x) := f(x) - b$, $x \in I^n$, also satisfies the assumptions of the Poincaré theorem. Therefore, there is $c \in I^n$ such that $f_b(c) = \mathbf{0}$ i.e., $f(c) = b$. Thus we have proved that $J^n \subset f(I^n)$. ■

Developing methods presenting here it is possible to give an elementary proof of the following Poincaré-Bolzano intermediate value theorem:

If a map $f = (f_1, \ldots, f_n) : I^n \to R^n, I := [-a, a]$, is a composition of two continuous maps $h : I^n \to X \subset R^n$, and $G : X \to R^n$, $f = g \circ h$, and if it satisfies the Bolzano condition:

$$f_i(x) \cdot f_i(y) < 0, \text{ for each } i \leq n \text{ and } x \in I_i^-, y \in I_i^+,$$

then $h(\partial I^n)$ *disconnets* R^n *and* $g^{-1}(\mathbf{0}) \cap \operatorname{Int} X \neq \emptyset$.

In the case when h is a homeomorphism and $g = h^{-1}$ we immediately obtain the Brouwer theorem on the invariance of a domain and a weak version of the Jordan separation theorem. When g is the identity map then we get Poincaré's theorem [8] from 1883, which for $n = 1$ is Bolzano's intermediate value theorem [1] from 1817; if $f(-a) \cdot f(a) < 0$, then f must assume zero.

There is no doubt that the sources of inspirations for Brouwer were Poincaré's works. Poincaré had formulated the problem of invariance of dimension, and in 1883 (see [11], pp. 368–370) used without proof the theorem on the invariance of domains in a proof in the theory of automorphic functions. Ideas of proofs of theorems of invariance of domain and dimension were suggested by Poincaré using separations "coupures" in his papers [9], [10] from 1903 and 1912.

Poincaré, finding himself under constant influx of a set ideas in the most diverse fields of mathematics and physics did not have time to be rigorous. His strong geometrical intuition allow him to ignore the pedantic strictness of proofs. He was often satisfied when his intuition gave him confidence that the proof of a theorem could carried out, then assigned the completion of the proof to others.

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