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A Note on Complex Unions of Subsets of the Real Line

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In this note we discuss the problem of measurability of complex unions $A + B$ of measurable subsets A, B of the real line. We show that some natural questions about this operation are undecidable within the theory ZFC. We also discuss the role of the countable chain condition of the standard boolean measure algebra in these considerations.

1. Introduction

We denote by $\mathbb{R}$ the real line and by $\mathbb{Q}$ the field of rational numbers. For sets $A, B \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ we consider the complex operations $A + B = \{a + b : a \in A \wedge b \in B\}$, $A + x = A + \{x\}$and $A \cdot B = \{a \cdot b : a \in A \wedge b \in B\}$ and so on. The σ-ideal of Lebesgue measure zero subsets of $\mathbb{R}$ is denoted by $\mathbb{L}$. By $\mathbb{K}$ we denote the σ-ideal of first category subsets of $\mathbb{R}$. We say that a set is nonmeasurable if it is nonmeasurable with respect to the Lebesgue measure. The family of all Lebesgue measurable subsets of $\mathbb{R}$ we denote by Leb. The classical Cantor subset of $\mathbb{R}$ we denote by $\mathbb{C}$, i.e. $\mathbb{C} = \{\sum_{n \geq 0} \frac{i_n}{3^n} : n \in \omega \wedge i_n \in \{0, 2\}\}$.

We shall work in ZFC set theory. By $\triangle$ we denote the symmetric difference of sets. The cardinality of a set X we denote by $|X|$. The cardinal number continuum is denoted by $\mathbf{c}$, i.e. $\mathbf{c} = 2^{\aleph_0}$. If X is an arbitrary set and κ is a cardinal number then $[X]^\kappa$ denotes the family of all subsets of X of cardinality κ. Let I be an arbitrary ideal of subsets of a set X. Then $add(I) = \min\{|S| : S \subseteq I \wedge \bigcup S \notin I\}$, $cov(I) = \min\{|S| : S \subseteq I \wedge \bigcup S = X\}$ and $non(I) = \min\{|T| : T \subseteq X \wedge T \notin I\}$.

We say that boolean algebra $\mathscr{B}$ satisfies the countable chain condition (c.c.c.) if each family of non-empty, pairwise disjoint elements of $\mathscr{B}$ is countable.

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We consider the field $\mathbb{R}$ as a linear space over the field $\mathbb{Q}$. Let us recall that any base of the space $\mathbb{R}$ over $\mathbb{Q}$ is called a *Hamel base*. For each $X \subseteq \mathbb{R}$ we denote by $Span(X)$ the linear subspace of $\mathbb{R}$ generated by the set X. If $X \subseteq \mathbb{R}$ and $n \in \omega$ then we put

$$Span(X, n) = \underbrace{\mathbb{Q} \cdot X + \ldots + \mathbb{Q} \cdot X}_{n}$$

Therefore for each set $X \subseteq \mathbb{R}$ we have $Span(X) = \bigcup\{Span(X, n) : n \in \omega\}$. Note that if U, V are linear subspaces of $\mathbb{R}$ and $U \cap V = \{0\}$ then $U + V$ is also a linear space and we denote this space by $U \oplus V$.

2. Nonmeasurable unions

Notice that $\mathbb{C} + \mathbb{C} = [0, 2]$. Let us consider any maximal, linearly independent over $\mathbb{Q}$ set $X \subseteq \mathbb{C}$. Then X is a Hamel base. Sierpiński observed (see [**6**]) that for some $n \in \omega$ the set $Span(X, n)$ is nonmeasurable. Indeed, since $\mathbb{R} = \bigcup_{n \in \omega} Span(X, n)$ there exists $m \in \omega$ such that $Span(X, m) \notin \mathbb{L}$. But there are $\mathbf{c}$ many pairwise disjoint translates of *Span(X, m)* (see e.g. Lemma 4), so the set $Span(X, m)$ is nonmeasurable.

Lemma 1. *Suppose that $non(\mathbb{L}) < cov(\mathbb{L})$. Let $\mathscr{A} \subseteq \mathbb{L}$ be an arbitrary family such that $\bigcup \mathscr{A} \notin \mathbb{L}$. Then there exists a subfamily $\mathscr{A}' \subseteq \mathscr{A}$ such that $\bigcup \mathscr{A} \notin Leb$.*

Proof. Let $A = \bigcup \mathscr{A}$. We may assume that $A \in Leb$. Since the Lebesgue measure is uniform we may find a nonmeasurable subset $T \subseteq A$ such that $|T| = non(\mathbb{L})$. For each $t \in T$ we fix some $A_t \in \mathscr{A}$ such that $t \in A_t$. Let $B = \bigcup\{A_t : t \in T\}$. Then $T \subseteq B$, so $B \notin \mathbb{L}$. If B were Lebesgue measurable then B would be a union of less than $cov(\mathbb{L})$ sets from the ideal $\mathbb{L}$, which contradicts the uniformity of Lebesgue measure. $\square$

A set $A \subseteq \mathbb{R}$ is an almost invariant set if $(\forall t \in \mathbb{R})(|(A + t) \triangle A| < \mathbf{c})$. The family of all almost invariant subsets of $\mathbb{R}$ is a σ-field. Sierpiński proved (see [**7**]) that if Continuum Hypothesis holds then there exists an almost invariant set $A \in \mathbb{L}$ such that $|A| = \mathbf{c}$. The same fact can be proved under the assumption $cov(\mathbb{K}) = \mathbf{c}$ (see [**3**] or [**4**]). Let us recall the following simple observation from [**3**]:

Lemma 2. *If $A \subseteq \mathbb{R}$ is an almost invariant set then for each $T \in [A]^{\mathbf{c}}$ we have $A - T = \mathbb{R}$.*

Proof. Suppose that $T \in [A]^{\mathbf{c}}$ and $b \in \mathbb{R} \setminus (A - T)$. Then $(\forall a \in A)(\forall t \in T)(b \neq a - t)$ so $(\forall a \in A)(\forall t \in T)(a \neq t + b)$ hence $A \cap (T + b) = \emptyset$, therefore $|(A + b) \setminus A| = \mathbf{c}$, hence A is not almost invariant. $\square$

Theorem 1. *The following sentence*

$$(\forall A \in \mathbb{L})(A + A \notin \mathbb{L} \rightarrow (\exists B \subseteq A)(A + B \notin Leb))$$

is independent from ZFC.

Proof. Suppose first that the inequality $non(\mathbb{L}) < cov(\mathbb{L})$ holds. Let $A \in \mathbb{L}$ be such that $A + A \notin \mathbb{L}$. Let us apply Lemma 1 to the family $\mathscr{A} = \{A + t : t \in A\}$. Then we obtain a set $B \subseteq A$ such that $\bigcup\{A + t : t \in B\} = A + B \notin Leb$.

Suppose now that $add(\mathbb{L}) = \mathbf{c}$. Then $cov(\mathbb{K}) = \mathbf{c}$ (see e.g. **[5]**). Therefore there exists an almost invariant set $C \in \mathbb{L}$ such that $|C| = \mathbf{c}$. Let $A = C \cup (-C)$. Then $A \in \mathbb{L}$ is almost invariant and $-A = A$. If $B \subseteq A$ and $|B| < \mathbf{c}$ then $A + B = \bigcup\{A + b : b \in B\} \in \mathbb{L}$. Suppose hence that $B \subseteq A$ and $|B| = \mathbf{c}$. Then $-B \subseteq -A = A$, therefore, by Lemma 2 we obtain $A - (-B) = A + B = \mathbb{R}$. Hence $(\forall B \subseteq A)(A + B \in Leb)$.

Finally let us note that both theories $ZFC \cup \{non(\mathbb{L}) < cov(\mathbb{L})\}$ and $ZFC \cup \{add(\mathbb{L}) = \mathbf{c}\}$ are equiconsistent with the theory ZFC (see e.g. **[1]** or **[2]**). □

Remark 1. *Recently Cichoń, Morayne and Ryll-Nardzewski proved in ZFC that there exists a subset $D \subseteq \mathbb{C}$ such that $\mathbb{C} + D \notin Leb$.*

3. The role of countable chain condition

In this section we consider arbitrary fields and ideals of subsets of the real line $\mathbb{R}$. We say that a family of sets $S \subseteq P(\mathbb{R})$ is *invariant* if for each $X \in S$, $x, y \in \mathbb{R}$ we have $x \cdot X + y \in S$. Notice that the σ-field of Borel subsets of $\mathbb{R}$, the σ-field of Lebesgue measurable sets and ideals $\mathbb{K}$, $\mathbb{L}$ are invariant.

Before we formulate the main result of this section we prove two technical lemmas.

Lemma 3. *Let $\mathscr{B}$ be a c.c.c. boolean algebra, $\{a_\xi\}_{\xi<\omega_1} \subseteq \mathscr{B}$ and $n \in \omega$. If $(\forall T \in [\omega_1]^n)(\prod_{\xi \in T} a_\xi = 0)$ then $|\{\xi \cdot a_\xi \neq 0\}| \leq \omega$.*

Proof. If $n = 1$ or $n = 2$ then the conclusion follows directly from c.c.c. of the algebra $\mathscr{B}$. Suppose hence that the lemma is true for $n \in \omega$ and let $\{a_\alpha\}_{\alpha<\omega_1} \in \mathscr{B}^{\omega_1}$ be such that $(\forall T \in [\omega_1]^{n+1})(\prod_{\xi \in T} a \ = 0)$ but $|\{\xi < \omega_1 : a_\xi \neq 0\}| = \omega_1$. For each $\alpha < \omega_1$ let $I_\alpha = \{\xi > \alpha : a_\alpha \cdot a_\xi \neq 0\}$. From the inductive assumption we deduce that $(\forall \alpha)(|I_\alpha| \leq \omega)$. And this allows us to build a subsequence $(a_{\gamma_\beta})_{\beta<\omega_1}$ of the sequence $\{a_\alpha\}_{\alpha<\omega_1}$ of nonzero pairwise disjoint elements of the algebra $\mathscr{B}$. □

Lemma 4. *Let $X \subseteq \mathbb{R}$ be linearly independent over $\mathbb{Q}$ and let $n \in \omega$. Then there are $\mathbf{c}$ pairwise disjoint translations of the set $Span(X, n)$.*

Proof. Let $X \subseteq \mathbb{R}$ be linearly independent over $\mathbb{Q}$, $n \in \omega$ and let us extend the set X to a Hamel base $H = \{h_\xi\}_{\xi<\mathbf{c}}$. Then $Span(X, n) \subseteq Span(H, n)$. Let $\{A_\alpha\}_{\alpha<\mathbf{c}} \subseteq$

$[\mathbf{c}]^{2n+1}$ be any family of pairwise disjoint sets and let $a_\alpha = \sum_{\xi \in A_\alpha} h_\xi$. It is easy to check that if $\alpha \neq \beta$ then $(Span(H, n) + a_\alpha) \cap (Span(H, n) + a_\beta) = \emptyset$. $\square$

Theorem 2. *Suppose that $\mathcal{S}$ is an invariant σ-field of subsets of $\mathbb{R}$, J is an invariant σ-ideal of subsets of $\mathbb{R}$ such that $J \subseteq \mathcal{S}$ and the quotient boolean algebra $\mathcal{S}/J$ satisfies the countable chain condition. Then the following three sentences are equivalent:*

1. $(\exists A \in J)(A + A \notin J)$,
2. $(\exists A \in J)(Span(A) \notin \mathcal{S})$,
3. $(\exists A \in J)(A + A \notin \mathcal{S})$.

Proof. 1) $\to$ 2) Suppose that $A \in J$ and $A + A \notin J$. Let B be a maximal linearly independent over $\mathbb{Q}$ subset of A. Then $B \in J$ and $Span(B) \notin J$. Let us choose two disjoint sets U and V such that $B = U \cup V$ and $|V| = \omega_1$. Then $Span(B) = Span(U) \oplus Span(V)$. If $Span(U) \notin \mathcal{S}$ then the proof of the first implication is done. Suppose hence that $Span(U) \in \mathcal{S}$. Then the assumption c.c.c. implies that $Span(U) \in J$.

We choose a Ulam-like matrix $\{V_{n,\xi} : n < \omega \wedge \xi < \omega_1\}$ on the set V with the following properties:

1. $(\forall n)(\forall \alpha, \beta < \omega_1)(V_{n,\alpha} \cap V_{n,\beta} = \emptyset)$,
2. $(\forall \xi < \omega_1)(|V \setminus \bigcup_{n \in \omega} V_{n,\xi}| \leq \omega$.

For each $n \in \omega$ we put $W_{n,\xi} = V_{0,\xi} \cup V_{1,\xi} \cup \ldots \cup V_{n,\xi}$. Property (2) implies that $(\forall \xi < \omega_1)(\exists n < \omega)(Span(U) \oplus Span(W_{n,\xi}) \notin J)$. Hence there exists a set $T \in [\omega_1]^{\omega_1}$ and $m < \omega$ such that $(\forall \xi \in T)(Span(U) \oplus Span(W_{m,\xi}) \notin J)$. Observe that $(\forall S \in [T]^{m+1})(\bigcap_{\xi \in S} W_{m,\xi} = \emptyset)$. But this easily implies that $(\forall S \in [T]^{m+1})(\bigcap_{\xi \in S} Span(W_{m,\xi}) = \{0\})$ and from this we deduce that $(\forall S \in [T]^{m+1})(\bigcap_{\xi \in S}(Span(U) \oplus Span(W_{m,\xi})) = Span(U))$. Therefore for each $S \in [T]^{m+1}$ we have $\bigcap_{\xi \in S}(Span(U) \oplus Span(W_{m,\xi})) \in J$. From Lemma 3 we get $\xi_0 \in T$ such that $Span(U) \oplus Span(W_{m,\xi_0}) \in J$ and this gives us a required contradiction.

2) $\to$ 3) Suppose now that $A \in J$ and $Span(A) \notin \mathcal{S}$. Let X be a maximal linearly independent over $\mathbb{Q}$ subset of A. Let $X_m = Span(X, m)$. The algebra $\mathcal{S}/J$ satisfies c.c.c. and Lemma 4 implies that there are $\mathbf{c}$ many pairwise disjoint translations of the set X_m. Therefore if $X_m \in \mathcal{S}$ then $X_m \in J$ for each $m \in \omega$. Notice that $Span(X) = Span(A) \notin \mathcal{S}$. Hence there exists $m \in \omega$ such that $X_m \notin \mathcal{S}$. Let $n \in \omega$ be minimal such number. Suppose that $n = 2k$. Then we have $X_k \in J$ and $X_k + X_k = X_n \notin \mathcal{S}$. Suppose hence that $n = 2k + 1$. Then $X_n \subseteq X_n + \mathbb{Q} \cdot X = X_{k+1} + X_{k+1}$. Then $X_{k+1} \in J$ and $X_n \notin J$ and this implies that $X_{k+1} + X_{k+1} \notin \mathcal{S}$.

The implication 3) $\to$ 1) is obvious. $\square$

Remark 2. *The assumption "$\mathcal{S}/J$ satisfies the countable chain condition" is necessary in the above theorem. Indeed, consider the σ-field $\mathcal{S} = P(\mathbb{R})$ and the σ-ideal $J = \mathbb{L}$. Then we have $\mathbb{C} \in J$ and $\mathbb{C} + \mathbb{C} \notin J$ and the sentence $(\forall A \in J)(A + A \in \mathcal{S})$ trivially holds.*

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