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On Nash Theorem

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The aim of this paper is to extend the Nash equilibrium theorem onto simplicial spaces.

1. Simplicial Structures

The main result of this paper is a theorem called here as Infimum Principle. As applications we derive some well-known results related to fixed points, minimax and equilibria theorems. We give another proofs for the most celebrated theorems in game theory, the Nash equilibrium theorem and the Gale-Nikaido theorem. Our study is based on and utilizes the techniques of simplicial structures and dual families. This approach enables us to derive not only classical theorems but also stimulates new research.

A collection $\{p_0, p_1, \ldots, p_n\} \subset \mathbf{R}^m$ of points of the m-dimensional Euclidean space $\mathbf{R}^m$ is said to be (affinely) independent if the vectors $p_1 - p_0, p_2 - p_0, \ldots, p_n - p_0$ are (linearly) independent.

Let $p_0, p_1, \ldots, p_n$ be independent points of the m-dimensional Euclidean space $\mathbf{R}^m$. The n-dimensional simplex $[p_0, p_1, \ldots, p_n]$ with vertices $p_0, p_1, \ldots, p_n$ is the subspace of $\mathbf{R}^n$ given by

$$\left\{x \in \mathbf{R}^m : x = \sum_{i=0}^{n} t_i p_i, t_i \geq 0 \textit{ for each } i = 1, 2, \ldots, n, \textit{ and } \sum_{i=0}^{n} t_i = 1\right\}.$$

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If $\{p_{i_0}, p_{i_1}, \dots, p_{i_k}\}$ is a subcollection of $k+1$ points of the collection $\{p_0, p_1, \dots, p_n\}$, then $[p_{i_0}, p_{i_1}, \dots, p_{i_k}] \subseteq [p_0, p_1, \dots, p_n]$ and the simplex $[p_{i_0}, p_{i_1}, \dots, p_{i_k}]$ is said to be a k-dimensional face of the simplex $[p_0, p_1, \dots, p_n]$.

The following Theorem on Indexed Families (due to W. Kulpa [7]) is our main tool in proving facts about dual families. We reprove it here for the reader's convenience.

Theorem 1. *(Theorem on Indexed Families). Let* $\sigma : [p_0, p_1, \dots, p_n] \to X$ *be a continuous function. For any covering* $U_0, U_1, \dots, U_n$ *of the subspace* $\sigma([p_0, p_1, \dots, p_n])$ *by non-empty open subsets of* X *there exists a non-empty subset of indices* $\{i_0, i_1, \dots, i_k\} \subseteq \{0, 1, \dots, n\}$ *such that* $\sigma([p_{i_0}, p_{i_1}, \dots, p_{i_k}]) \cap U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k} \neq \emptyset$.

Proof. For $i = 0, 1, \dots, k$, let d_i be a function on the simplex $[p_0, p_1, \dots, p_n]$ given by

$$d_i(x) = d(x, [p_0, p_1, \dots, p_n] \setminus \sigma^{-1}(U_i)),$$

where $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$ is the distance between the point x and the subset Y in $\mathbf{R^m}$. Each of the functions d_i is continuous and since the sets $\sigma^{-1}(U_i)$ are open,

$$d_i(x) = 0 \textit{ if and only if } x \neq \sigma^{-1}(U_i).$$

The function f given by

$$f(x) = \sum_{i=0}^{n} \left(\frac{d_i(x)}{\sum_{i=0}^{n} d_i(x)} \right) p_i$$

is a continuous function defined on the simplex $[p_0, p_1, \dots, p_n]$ into $[p_0, p_1, \dots, p_n]$. According to the Brouwer Fixed Point Theorem, there exits $a \in [p_0, p_1, \dots, p_n]$ such that $f(a) = a$. Thus

$$a = \sum_{i=0}^{n} \left(\frac{d_i(a)}{\sum_{i=0}^{n} d_i(a)} \right) p_i . \tag{1}$$

Let $\{i_0, i_1, \dots, i_k\}$ be the set of all indices i such that

$$\frac{d_i(a)}{\sum_{i=0}^{n} d_i(a)} \neq 0 . \tag{2}$$

From (1), $a \in [p_{i_0}, p_{i_1}, \dots, p_{i_k}]$. From (2),

$$i \in \{i_0, i_1, \dots, i_k\} \textit{ if and only if } a \in \sigma^{-1}(U_i).$$

Subsequently,

$$\sigma(a) \in \sigma([p_{i_0}, p_{i_1}, \dots, p_{i_k}]) \cap U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k}. \qquad \square$$

In the n-dimensional Euclidean space $\mathbf{R^n}$, let the points $e_0, e_1, \dots, e_n$ be given by:

$$e_0 = (0, 0, \dots, 0) \textit{ and } e_i = (0, \dots, 0, \overset{ith}{1}, 0, \dots, 0) \textit{ for each } i = 1, \dots, n.$$

The $(n+1)$ points are affinely independent and the simplex $[e_0, e_1, \ldots, e_n]$ is going to be called the standard $n-$dimensional simplex and denoted by Δ_n. Each $n-$dimensional simplex $[p_0, p_1, \ldots, p_n]$ is affinely isomorphic to the $n-$dimensional standard simplex via the isomorphism E given by:

(3) $$E\left(\sum_{i=0}^{n} t_i e_i\right) = \sum_{i=0}^{n} t_i p_i .$$

By abusing slightly preciseness of formal exposition but by gaining some clarity in defining simplicial structures (see Definition (ii), below), we are going to identify any k-dimensional standard simplex $[e_0, e_1, \ldots, e_k]$ via the affine isomorphism E (see 3).

Let X be a topological space. The term singular simplex in X is coined to mean any continuous map σ from a standard simplex into the space X. The collection of all singular simplexes in X is denoted by $\sum(X)$.

A collection $\mathscr{S}$ of singular simplexes in X is called a *simplicial structure* on X if: (i) For any finite subset $\{a_0, a_1, \ldots, a_n\}$ of (not necessarily distinct) points of the space X there exists $\sigma \in \mathscr{S}$ such that $\sigma : [e_0, e_1, \ldots, e_n] \to X$ and $\sigma(e_i) = a_i$ for each $i = 0, 1, \ldots, n$; (ii) If $\sigma \in \mathscr{S}$, then the restriction of σ to any face of the domain of σ belongs to $\mathscr{S}$.

A topological space X together with a simplicial structure $\mathscr{S}$ on X is going to be referred to as a *simplicial space*.

Let $(X, \mathscr{S})$ be a simplicial space. We say that $A \subseteq X$ is a *simplicially convex subset* of X if for each finite subset $\{a_0, a_1, \ldots, a_n\}$ of the set A and for each $\sigma \in \mathscr{S}$ such that $\sigma(e_i) = a_i$ for each $i = 0, 1, \ldots, n$, the set $\sigma(\Delta_n)$ is contained in A.

Example. Let X be a linear topological space. Define $\mathscr{S} = \mathscr{A}(X)$ to be the family of all affine maps, $\sigma : [e_0, e_1, \ldots, e_n] \to X$, given by

$$\sigma\left(\sum_{i=0}^{n} t_i e_i\right) = \sum_{i=0}^{n} t_i \sigma(e_i).$$

Convex subsets of the simplicial space $(X, \mathscr{A}(X))$ coincide with convex subsets of the linear space X. **It is tacitly assumed (unless otherwise stated) that a linear space X is a simplicial space with the simplicial structure of all affine maps.**

Example. Let $\mathscr{S}$ be a simplicial structure on X and let $f : X \to Y$ be a continuous surjection. Then $f(\mathscr{S})$ given by

$$f(\mathscr{S}) = \{f \circ \sigma : \sigma \in \mathscr{S}\}$$

is a simplicial structure on Y. It is easy to verify that if $C \subseteq Y$ and $f^{1}(C)$ is a convex subset of the simplicial space $(X, \mathscr{S})$, then C is a convex subset of the simplicial space $(Y, f(\mathscr{S}))$.

Let $(X_1, \mathscr{S}_1)$ and $(X_2, \mathscr{S}_2)$ be simplicial spaces. The *product of the two simplicial spaces* is the space $(X, \mathscr{S})$ where:

$$X = X_1 \times X_2$$

and

$$\mathscr{S} = \{\sigma : \exists_n \exists_{\sigma_1 \in \mathscr{S}_1} \exists_{\sigma_2 \in \mathscr{S}_2} \sigma_1 : \Delta_n \to X_1, \sigma_2 : \Delta_n \to X_2, \textit{ and } \sigma = (\sigma_1, \sigma_2)\}.$$

Lemma 1. *If $(X_1, \mathscr{S}_1)$ and $(X_2, \mathscr{S}_2)$ are simplicial spaces, then their product $(X, \mathscr{S})$ is also a simplicial space. Moreover, if A is a convex subset of the simplicial space $(X_1, \mathscr{S}_1)$ and B is a convex subset of the simplicial space $(X_2, \mathscr{S}_2)$, then $A \times B$ is a convex subset of the simplicial space $(X, \mathscr{S})$.*

2. Dual Families

For given two sets X and Y, let $\mathscr{F} = \{F(x) : x \in X\}$ be a family of non-empty subsets of Y indexed by the elements of the set X. Such a family gives rise to a *dual family*, $\mathscr{F}'$, of subsets of the set X indexed by elements of Y, defined as follows:

$$\mathscr{F}' = \{F'(y) : y \in Y\},$$

where

$$F'(y) = \{x \in X : y \in F(x)\}.$$[1]

We have the following duality:

$$y \in F(x) \textit{ if and only if } x \in F'(y). \tag{4}$$

Consequently, we get the following.

Observation. *If the family $\mathscr{F}$ consists of non-empty sets, then the dual family $\mathscr{F}'$ is a covering of X.*

Theorem 2. *Let X be a simplicial space and let $h : X \to Y$ be a continuous map into the space Y such that $\overline{h(X)}$ is a compact subset of Y. Let $\mathscr{F} = \{F(y) : y \in Y\}$ be a family of non-empty convex subsets of the space X such that $F'(x)$ is an open subset of Y for each $x \in Y$. Then there exists a point $a \in X$ such that*

$$a \in F(h(a)).$$

Proof. Since $\overline{h(X)}$ is compact, there are points $x_0, x_1, \dots, x_m \in X$ such that

$$h(X) \subseteq F'(x_0) \cup F'(x_1) \cup \dots \cup F'(x_m).$$

[1] The definition of families $\{F(x) : x \in X\}$ and their duals can also be given in terms of set-valued mappings $F : X \to 2^Y$ or in terms of subsets of the product $X \times Y$. For instance, if F was considered as a map $F : X \to 2^Y$, then $F' : Y \to 2^X$ and F' would be a kind of inverse map to F. For our exposition, we prefer that both F and F' be families of sets.

Let us take a singular simplex from the simplicial structure of X, $\sigma : [e_0, e_1, \dots, e_m] \to X$, such that $\sigma(e_i) = x_i$ for each $i = 0, 1, \dots, m$. From the Theorem on Indexed Families (Theorem 1),

$$\sigma([e_{i_0}, e_{i_1}, \dots, e_{i_k}]) \cap h^{-1}(F'(x_{i_0})) \cap h^{-1}(F'(x_{i_1})) \cap \dots \cap h^{-1}(F'(x_{i_k})) \neq \emptyset$$

for some $\emptyset \neq \{i_0, i_1, \dots, i_k\} \subseteq \{1,2, \dots, n\}$. Let $a \in \sigma([e_{i_0}, e_{i_1}, \dots, e_{i_k}]) \cap h^{-1}(F'(x_{i_0})) \cap h^{-1}(F'(x_{i_1})) \cap \dots \cap h^{-1}(F'(x_{i_k}))$. Hence $h(a) \in F'(x_{i_0}) \cap F'(x_{i_1}) \cap \dots \cap F'(x_{i_k})$. It follows from (4) that

$$\{x_{i_0}, x_{i_1}, \dots, x_{i_k}\} \subseteq F(h(a)).$$

Since $F(h(a))$ is a convex subset of X,

$$\sigma([e_{i_0}, e_{i_1}, \dots, e_{i_k}]) \subseteq F(h(a)).$$

Since $a \in \sigma([e_{i_0}, e_{i_1}, \dots, e_{i_k}])$,

$$a \in F(h(a)).$$ □

The following lemma is simple and probably well known. Since it is indispensable for us, we give its short proof.

Lemma 2. *Let $f : X \times Y \to \mathbf{R}$ be a continuous real function defined on the product of two compact spaces. Then the real functions $g : X \to \mathbf{R}$ given by*

$$g(x) = \sup \{f(x, y) : y \in Y\}$$

and $h : Y \to \mathbf{R}$ given by

$$h(y) = \inf \{f(x, y) : x \in X\}$$

are continuous.

Proof. Let (a, b) be an open interval in $\mathbf{R}$ and let $c = g(x_0) \in (a, b)$. Since Y is compact and $f(\{x_0\} \times Y) \subseteq (-\infty, b)$, there exists an open neighborhood U of x_0 such that $f(U \times Y) \subseteq (-\infty, b)$. Let $y_0 \in Y$ satisfy that $f(x_0, y_0) = c$. Let V be an open neighborhood of x_0 such that $f(V \times \{y_0\}) \subseteq (a, b)$. Hence $g(U \cap V) \subseteq (a, b)$.
To show continuity of the function h use the fact that $\inf(-f) = -\sup f$. □

Let X be a simplicial space. A function $f : X \times Y \to \mathbf{R}$ is said to be *quasi-convex with respect to the first variable* x if for each $y \in Y$ and $r \in R$ the set $\{x \in X : f(x, y) < r\}$ is simplicially convex, and f is said to be *quasi-concave* if $(-f)$ is quasi-convex.

3. Infimum Principle

Theorem 3. *(Infimum Principle). Let $g_s : X \times Y \to R$, $s \in S$, be a family of continuous functions from a product of a compact simplicial space X and a topological space Y such that;*

1. *each of the functions g_s is quasi-convex with respect to the first variable x,*
2. *for each finite subset $S_0 \subset S$ and for each point $y \in Y$ there is a point $a \in X$ with*

$$g_s(a, y) = \inf_{x \in X} g_s(x, y) \text{ for each } s \in S_0 .$$

Then for each continuous map $h : X \to Y$ there is a point $a \in X$ such that

$$g_s(a, h(a)) = \inf_{x \in X} g_s(x, h(a)) \text{ for each } s \in S .$$

Proof. Fix a finite set of functions $g_1, \ldots, g_n$ from the family $\{g_s : s \in S\}$. For each $i = 1, 2, \ldots, n$, the function $\overline{g_i}$ is given by

$$\overline{g_i}(y) = \inf_{x \in X} g_i(x, y) .$$

By Lemma 2, the functions $\overline{g_i}$ are continuous. Let $\varepsilon > 0$ be given. For each $y \in Y$ and for each $i = 1, \ldots, n$, the set $A_i(y)$ is given by

$$A_i(y) = \{x \in X : g_i(x, y) < \overline{g_i}(y) + \varepsilon\}.$$

By 1., $A_i(y)$ is convex for each $y \in Y$ and $i = 1, \ldots, n$.

By continuity of g_i's, the dual sets

$$A_i'(x) = \{y \in Y : g_i(x, y) < \overline{g_i}(y) + \varepsilon\}$$

are open for each $x \in X$ and $i = 1, \ldots, n$.

By 2., appealing to the definition of $\overline{g_i}(y)$ there exists $a \in X$ such that $g_i(a, y) < \overline{g_i}(y) + \varepsilon$ for each $i = 1, \ldots, n$, which means that $a \in A(y) := \bigcap\{A_i(y) : i = 1, 2, \ldots, n\}$.

Setting $\mathscr{F} = \{A(y) \colon Y \in X\}$ we get a family of non-empty convex subsets of simplicial space indexed by elements $y \in X$ such that the dual set $A'(x)$ is open for each $x \in X$.

Theorem 2 applied to the family $\mathscr{F}$ and to the function on $h : X \to Y$ yields a point $a_\varepsilon \in X$ such that $a_\varepsilon \in A(h(a_\varepsilon)) = \bigcap\{A_i(h(a_\varepsilon)) : i = 1, 2, \ldots, n\}$. Hence, for each $i = 1, \ldots, n$, $g_i(a_\varepsilon, h(a_\varepsilon)) < \overline{g_i}(h(a_\varepsilon)) + \varepsilon$.

For a given $\varepsilon > 0$, we set

$$K(\varepsilon) = \{x \in X : g_i(x, h(x)) - \overline{g_i}(h(x)) \leq \varepsilon \text{ for each } i = 1, \ldots, n\}.$$

We just showed that the sets $K(\varepsilon)$ are non-empty for each $\varepsilon > 0$, and since each of the functions g_i and $\overline{g_i}$ is continuous, the sets $K(\varepsilon)$ are also closed. By compactness of the space X, there exists a point $a \in X$ such that $a \in K(\varepsilon)$ for each $\varepsilon > 0$. Thus we have just proved that for each finite set $S_0 \subset S$ the set

$$K(S_0) := \{a \in X : g_s(a, h(a)) = \inf_{x \in X} g_s(x, h(a)) \text{ for each } s \in S_0$$

is non-empty. Applying once again compactness argument to the centered family of compact set $\{K(S_0) : S_0$ finite subset of $S\}$ we infer that $K(S) = \bigcap\{K(S_0) : S_0$ finite subset of $S\}$ is non-empty too. This completes the proof. $\square$

Applying the Helly Theorem to families of convex subsets $\{A(y): y \in Y\}$ (see [8]) in the previous proof we can get the following version of the Infimum Principle of the Helly type.

Theorem 4. *Helly Infimum Principle. Let $g_s : X \times Y \to R$, $s \in S$, be a family of continuous functions from a product of a compact simplicial space X of covering dimension less than n, dim $X < n$, and a topological space Y such that;*

1. each of the functions g_s is quasi-convex with respect to the first variable x,

2. for each finite subset $S_0 \subset S$ of cardinality not greater than n and for each point $y \in Y$ there is a point $a \in X$ with

$$g_s(a, y) = \inf_{x \in X} g_s(x, y) \text{ for each } s \in S_0.$$

Then for each continuous map $h : X \to Y$ there is a point $a \in X$ such that

$$g_s(a, h(a)) = \inf_{x \in X} g_s(x, h(a)) \text{ for each } s \in S.$$

Remark. In the version of our Infimum Theorem with conclusions inf one could get more versions of the theorem specified as: (1) sup/inf, (2) inf/sup, and (3) sup/sup. To state and to prove one of the new versions, one would have to replace quasi-convexity by quasi-concavity in part (2) if supremum is involved and make appropriate adjustment in the original proof utilizing that $\inf(-g) = -\sup g$.

4. Consequences

Theorem 5. *Let X be a compact simplicial space and let $g_s : X \times X \to [0, \infty)$, $s \in S$, be a family of continuous functions quasi-convex with respect to the first variable such that;*

1. for each $x \in X$ and $s \in S$, $g_s(x, x) = 0$,

2. for each two distinct points $x, y \in X$ there is $s \in S$ with $g_s(x, y) > 0$.

Then any continuous map $h : X \to X$ has a fixed point.

Proof. According to Infimum Principle there is a point $a \in X$ such that $g_s(a, h(a)) = \inf_{x \in X} g_s(x, h(a))$ for each $s \in S$. From the assumption it follows that $a = h(a)$. □

Remark. If X is a subset of normed space then the family consisting of one function $g(x, y) = \|x - y\|$ realizes the assumptions of the theorem and in this case we obtain the Schauder Fixed Point Theorem for convex compact subsets of normed space.

If X is a convex compact subset of locally convex linear space then the family of all continuous seminorms realizes the assumptions of the theorem and in this case we obtain the Schauder-Tychonoff Fixed Point Theorem.

Example. Fix $n > 1$ and let us set $X := Q \cup \{p\}$, where $Q := \{x \in R^n : \|x\| < 1\}$ and $p := (1, 0, \dots, 0) \in R^n$. The set X is convex subset of R^n. Let $\mathscr{L}$ be the affine simplicial structure consisting of all the affine simplices in X. Describe a new topology $\mathscr{T}$ on X generated by a base of open neighbourhoods; for every $x \in Q$ and ε define neighbourhoods $U_x(\varepsilon) := \{y \in Q : \|x - y\| < \varepsilon\}$ the same as in Euclidean topology, and for p, put $U_p(\varepsilon) := \{p\} \cup \{x \in Q : \|x\| > \varepsilon\}, 0 < \varepsilon < 1, \varepsilon \to 1$.

The topology $\mathscr{T}$ is weaker than the Euclidean topology on X and therefore the triple $(X, \mathscr{T}, \mathscr{L})$ is a topological simplicial space. The space X is locally convex at each point $x \neq p$ because the neighbourhoods $U_x(\varepsilon)$ are linearly convex. It is easy to see that X is not locally convex at p because for each point $x \in U_p(\varepsilon)$, $x \neq p$, the 1-dimensional simplex with vertices x and $-x$ must contain $0 = (0, \dots, 0)$.

It is known that the quotient space $Q/\partial Q$ is homeomorphic to n-dimensional sphere $S^n = \{x \in R^{n+1} : \|x\| = 1\}$, and therefore the space has not fixed point property.

Let $h : X \to S^n$ be a homeomorphism and define a metric $g : X \times X \to [0, \infty)$ inducing the topology on X by

$$g(x, y) := \|h(x) - h(y)\| \text{ for each } x, y \in X$$

Observe that for each r the set $\{x \in X : g(x, y) < r\}$ is convex for each $y \neq p$ and it is not convex for $y = p$.

This example shows that the assumption on convexity with respect to the first variable can not be omitted. It also shows that the assumption on local convexity in the Schauder fixed point theorem is essential. But this does not solve the Schauder Problem from the *The Scottish Book* ([11], Problem 54) to simplicial affine structures with linear topology (in our example X has simplicial affine structure with non-linear topology).

It is possible to obtain some kinds theorems of Ky Fan type for example (see [2]);

Theorem 6. *(Ky Fan). Let $h : X \to Y$ be a continuous map from a convex compact subset X of a normed space Y. Then there is a point $a \in X$ such that*

$$\|a - h(a)\| = \inf_{x \in X} \|x - h(a)\|.$$

Proof. Apply Infimum Principle to function $g(x, y) = \|x - y\|$. □

Theorem 7. *(Ky Fan Minimax Inequality). Let $g : X \times X \to \mathbf{R}$ be a continuous function quasi-concave with respect to the first variable x, where X is a compact simpicial space.*

Then the following inequality holds

$$\inf_{y \in X} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} g(x, x)$$

Proof. Applying the Infimum Theorem to the map $(-g)$ and the identity map $h(x) = x$ we obtain a point $a \in X$ such that

$$g(a, a) = \sup_{x \in X} g(x, a)$$

Hence

$$\inf_{y \in X} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} g(x, a) \leq g(a, a) = \sup_{x \in X} g(x, x) \qquad \square$$

In fact the second theorem of Ky Fan is a simple consequences of the following minimax inequality.

Theorem 8. *(Minimax Inequality). Let $g_s : X \times Y \to R$, $s \in S$, be a family of continuous functions from a product of a compact simplicial space X and a topological space Y such that;*

1. *each of the functions g_s is quasi-concave with respect to the first variable x,*
2. *for each finite subset $S_0 \subset S$ and for each point $y \in Y$ there is a point $a \in X$ with*

$$g(a, y) = \sup_{x \in X} g(x, y) \text{ for each } s \in S_0.$$

Then for each continuous map $h : X \to Y$ there is a point $a \in X$ such that

$$\inf_{y \in Y} \sup_{x \in X} g(x, y) \leq g_s(a, h(a)) \text{ for each } s \in S.$$

Proof. Since $\sup g = -\inf(-g)$ from the Infimum Principle we infer that there is a point $a \in X$ such that

$$g(a, h(a)) = \sup_{x \in X} g(x, h(a))$$

Hence

$$\inf_{y \in Y} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} g_s(x, h(x)) = g(a, h(a)). \qquad \square$$

Let $X := \Pi\{X_s : s \in S\}$ be a Cartesian product of simplicial spaces. For each $i \in S$ let define the Nash projection $N_i : X \times X \to X$;

$$[N_i(x, y)]_s := \begin{cases} y_s & \text{if } s \neq i, \\ x_i & \text{if } s = i \end{cases}$$

where z_s means the s-th coordinate of a point $z \in X$. It is clear that $N_i(x, x) = x$.

Theorem 9. *Let be given 1. $X = \Pi\{X_s : s \in S\}$ a Cartesian product of compact simplicial spaces, 2. non-empty sets T_s, $s \in S$, of indices, and 3. continuous functions $f_{ts} : X \to \mathbf{R}$, $s \in S$ and $t \in T_s$, quasi-convex with respect to the variable x_s and such that for each $y \in$ and $s \in S$ there is a point $a \in X$ with*

$$f_{ts}(N_s(a, y)) = \inf_{x \in X} f_{ts}(N_s(x, y)) \text{ for each } t \in T_s$$

Then there exists a point $a \in X$ such that

$$f_{ts}(a) = \inf_{x \in X} (f_{ts}(N_s(x, a)).$$

Proof. Let $g_{ts}(x, y) := f_{ts}(N_s(x, y))$. Fix $y \in X$. From the assumptions it follows that for each $s \in S$ there is a $a^s \in X$ such that for each $t \in T_s$, g_{ts}^y assumes infimum in the point a^s; $g_{ts}(a^s, y) = \inf_{x \in X} g_{ts}(x, y)$.

Let $a \in X$ be the unique point such that the s-th coordinate of a is equal the s-th coordinate of a^s; $a_s = (a^s)_s$. From definition of the Nash projections N_s it follows that for each $s \in S$, $N_s(a, y) = N_s(a^s, y)$, and in consequence $g_{ts}(a, y) = \inf_{x \in X} g_{ts}(x, y)$ for each $s \in S$.

We have just showed that the assumptions of the Infimum Theorem hold. Applying the theorem to the identity map, $h(x) = x$, and having in mind that $N_s(x, x) = x$, we infer that there is a point $a \in X$ such that for each $s \in S$,

$$f_{ts}(a) = g_{ts}(a, a) = \inf_{x \in X} f_{ts}(N_s(x, a)) \qquad \square$$

Theorem 10. *(Equilibrium Theorem). Let $f_s : X \to \mathbf{R}, s \in S$, be a family of continuous maps from a Cartesian product $X = \Pi\{X_s : s \in S\}$ of compact simplicial spaces such that each composition $f_s \circ N_s : X \times X \to \mathbf{R}$ is a quasi-convex function with respect to first variable x. Then there exists a point $a \in X$ such that*

$$f_s(a) = \inf_{x \in X} (f_s \circ N_s)(x, a).$$

Proof. From compactness of X and continuity of functions $g_s(x, y) := f_s(N_s(x, y))$ it follows that for each $y \in Y$ and $s \in S$ there is a point $a^s \in X$ such that g_s^y assumes infimum in the point a^s; $g_s(a^s, y) = \inf_{x \in X} g_s(x, y)$.

Next, apply the above theorem to one-point sets $T_s = \{s\}$. $\square$

Theorem 11. *(Nash Equilibrium Theorem). Let $f_i : X \to \mathbf{R}, i = 1, \ldots, n$, be a family of continuous maps from a Cartesian product $X = X_1 \times \ldots \times X_n$ of compact simplicial spaces such that each function $f_i : X \to \mathbf{R}$ is a quasi-concave with respect to i-th variable $x_i \in X_i$. Then there exists a point $a \in X$ such that*

$$f_i(a) = \sup_{x_i \in X_i} (f_i(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)).$$

Theorem 12. *(Minimax Version.) Let $h_t : X_1 \times X_2 \to \mathbf{R}, t \in T$, be a family of continuous functions from a Cartesian product of two simplicial compact spaces and such that each of them is quasi-concave with respect to the first variable x_1 and quasi-convex with respect to the second variable x_2 and assume that for each point $(x_1, x_2) \in X_1 \times X_2$ there is a point $(a_1, a_2) \in X_1 \times X_2$ such that for each $t \in T$,*

$$h_t(x_1, a_2) = \inf_{x_2 \in X_2} h_t(x_1, x_2) \quad \text{and} \quad h_t(a_1, x_2) = \sup_{x_1 \in X_1} h_t(x_1, x_2).$$

Then there is a point $(a_1, a_2) \in X_1 \times X_2$ such that for each $t \in T$,

$$h_t(a_1, a_2) = \inf_{x_2 \in X_2} h_t(a_1, x_2) \quad \text{and} \quad h_t(a_1, a_2) = \sup_{x_1 \in X_1} h_t(x_1, a_2).$$

Proof. Let $S := \{1,2\}$ and $T_i := T$ for each $i = 1, 2$. Define

$$f_{ti}(x_1, x_2) := \begin{cases} -h_t(x_1, x_2) & \text{if } i = 1 \\ h_t(x_1, x_2) & \text{if } i = 2 \end{cases}$$

According to Equilibrium Theorem there is a point $(a_1, a_2) \in X_1 \times X_2$ such that

$$-h_t(a_1, a_2) = \inf_{x_1 \in X_1} (-h_t)(x_1, a_2) \quad \text{and} \quad h_t(a_1, a_2) = \inf_{x_2 \in X_2} h_t(a_1, x_2)$$

Since $-\inf(-h) = \sup(h)$ we see that this theorem is a simple consequence of the previous theorem. □

Theorem 13. *(von Neumann Minimax Principle). Let X and Y be compact simplicial spaces and let $h_t : X \times Y \to \mathbf{R}$, $t \in T$, be continuous functions. Suppose further that each of the functions h_t is quasi-concave with respect to the first variable x and quasi-convex with respect to the second variable y and assume that for each point $(x, y) \in X \times Y$ there is a point $(a, b) \in X \times Y$ such that for each $t \in T$,*

$$h_t(x, b) = \inf_{y \in Y} h_t(x, y) \quad \textit{and} \quad h_t(a, y) = \sup_{x \in X} h_t(x, y).$$

Then there is a point $(a, b) \in X \times Y$ such that for each $t \in T$;

$$h_t(a, b) = \max_{x \in X} \min_{y \in Y} h_t(x, y) = \min_{y \in Y} \max_{x \in X} h_t(x, y).$$

Proof. The following inequality holds true for arbitrary function, in particular for every function h_t:

$$\sup_{x \in X} \inf_{y \in Y} h_t(x, y) \leq \inf_{y \in Y} \sup_{x \in X} h_t(x, y).$$

We can apply our Theorem to obtain

$$h_t(a, b) = \inf_{y \in Y} h_t(a, y) \leq \sup_{x \in X} \inf_{y \in Y} h_t(x, y) \leq \inf_{y \in Y} \sup_{x \in X} h_t(x, y) \leq \sup_{x \in X} h_t(x, b) = h_t(a, b).$$

Because of compactness of X and Y and continuity of the functions h_t, both $\max_{x \in X} \min_{y \ Y} h_t(x, y)$ and $\min_{y \in Y} \max_{x \in X} h_t(x, y)$ exist. Hence

$$\max_{x \in X} \min_{y \in Y} h_t(x, y) = \min_{y \in Y} \max_{x \in X} h_t(x, y) = h_t(a, b) \textit{ for each } t \in T. \qquad \square$$

5. Limit Set-Valued Maps

A set-valued map $H : X \to 2^Y$, where X and Y are metric spaces, is called a *subupper limit set-valued map* if there is a family $\{h_n : X \to Y \mid n = 1, 2, \ldots\}$ of continuous maps such that;

If $\lim_{k\to\infty}(x_{n_k}, f_{n_k}(x_{n_k})) = (x, y)$, then $y \in H(x)$.
The family $h_n : n = 1, \dots, n$ is said to be basic for H.
Denote by $\underset{n\to\infty}{Ls}\ h_n : X \to 2^Y$;

$$\left(\underset{n\to\infty}{Ls}\ h_n\right)(x) := \left\{y \in Y : y = \lim_{n\to\infty} h_{n_k}(x_{n_k}) \text{ for some } x_{n_k} \underset{n\to\infty}{\to} x\right\}$$

Example. Let $D = \{d_0, d_1, \dots\}$ be a countable subset of Y. For each $n \in N$ set $h_n : X \to Y$ to be a constant map; $h_n(x) = d_n$ for each $x \in X$. Then $H := \underset{n\to\infty}{Ls}\ h_n$ is a set-valued map $X \to 2^Y$ such that $H(x) = \bar{D}$ for each $x \in X$.

Theorem 14. *If $H : X \to 2^X$ is a subupper limit set-valued map defined on compact metric space X with fixed point property then there is $a \in X$ such that $a \in H(a)$.*

Proof. Let $\{h_n : X \to X \mid n = 1, 2, \dots\}$ be a basic family for the map H. Since X has a fixed point property for each n there is a point $a_n \in X$ such that $h(a_n) = a_n$. By compactness of X, the sequence $\{(a_n, h(a_n)) : n = 1, 2, \dots\}$ contains a covergent subsequence, say

$$\lim_{k\to\infty} (a_{n_k}, h_{n_k}(a_{n_k})) = (a, b).$$

It follows that $a = b$ and $a \in H(a)$. □

A set-valued map $H : X \to 2^Y$ is said to be *upper semicontinuous* if $H\ ^1(V) = \{x \in X : H(x) \subset V\}$ is an open set in X provided that V is open in Y.

Theorem 15. *Let X be a compact convex subspace of a normed space. If $H : X \to 2^X$ is upper semicontinuous and $H(x)$ is non-empty convex compact set for each $x \in X$, then H is a subupper limit set-valued map.*

Proof. To define a function h_n from a sequence that witnesses subupper limit set-valuedness of H, fix ε, $0 < \varepsilon < \frac{1}{n}$. Let $U(x, \varepsilon)$ be given by

$$U(x, \varepsilon) = \{y \in X : H(y) \subset B(H(x), \varepsilon)\} \cap B(x, \varepsilon).$$

By compactness of X, the open covering $\{U(x, \varepsilon) : x \in X\}$ has a finite starrefinement $\{V_0(\varepsilon), V_1(\varepsilon), \dots, V_m(\varepsilon)\}$, i.e., for each $x \in X$ there exists $\bar{x} \in X$ such that $\bigcup\{V_i(\varepsilon) : x \in V_i(\varepsilon)\} \subset U(\bar{x}, \varepsilon)$.

For each $i = 1, \dots, m$ let p_i be arbitrary point of the set $H(V_i(\varepsilon)) = \bigcup\{H(x) : x \in V_i(\varepsilon)\}$. We set

$$h_n(x) := \sum_{i=0}^{n} \left(\frac{d_i(x)}{\sum_{i=0}^{n} d_i(x)}\right) p_i,$$

where $d_i(x) = d(x, X \setminus V_i(\varepsilon))$. The function $h_n : X \to X$ is continuous.

For a given $x \in X$, if $\bar{x} \in X$ is such that $\bigcup\{V_i(\varepsilon) : x \in V_i(\varepsilon)\} \subset U(\bar{x}, \varepsilon)$, then $p_i \in B(H(\bar{x}), \varepsilon)$ whenever $x \in V_i(\varepsilon)$. Since $x \in V_i(\varepsilon)$ if and only if $d_i(x) \neq 0$,

$p_i \in B(H(\bar{x}), \varepsilon)$ whenever $d_i(x) \neq 0$. Since $B(H(\bar{x}), \varepsilon)$ is convex, $h_n(x) := \sum_{i}^{n}{}_0 [(d_i(x))/(\sum_{i}^{n}{}_0 d_i(x))]\, p_i \in B(H(\bar{x}), \varepsilon)$. Thus we have proved that for each x there is $\bar{x}$ such that

$$|x - \bar{x}\| < \varepsilon \quad \text{and} \quad d(h_n(x), H(\bar{x})) < \varepsilon$$

We shall prove that the sequence $\{h_n : n = 1, 2, \ldots\}$ is basic for H. Towards this end, assume that $\lim_{k\to\infty} (x_{n_k}, h_{n_k}(x_{n_k})) = (x, y)$. We have just showed that for each x_{n_k} there is $\overline{x_{n_k}}$ such that $\|x_{n_k} - \overline{x_{n_k}}\| < \frac{1}{nk}$ and $d(h_{n_k}(x_{n_k}), H(\overline{x_{n_k}})) < \frac{1}{nk}$. The latter means that there exists $\overline{y_{n_k}} \in H(\overline{x_{n_k}})$ such that $\|h_{n_k}(x_{n_k}) - \overline{y_{n_k}}\| < \frac{1}{nk}$. Hence $\lim_{k\to\infty} (x_{n_k}, \overline{y_{n_k}}) = (x, y)$. Since H is upper semi-continuous, $y \in H(x)$. □

Corollary *(Kakutani's Fixed Point Theorem). Let $H : X \to 2^X$ be an upper semicontinuous map on a compact subspace X of a normed space. If $H(x)$ is a non-empty closed and convex subset of X for each $x \in X$, then the map H has a fixed point, i.e., there exists a point $a \in X$ such that $a \in H(a)$.*

More informations on extensions of Kakutani's Theorem the reader will find in [9] and [19].

Theorem 16. *(Infimum Theorem for Multi-Valued Maps). Let $g_s : X \times Y \to R$, $s \in S$, be a family of continuous functions from a product of a compact metric simplicial space X and a compact metric space Y such that;*

1. *each of the functions g_s is quasi-convex with respect to the first variable x,*
2. *for each finite subset $S_0 \subset S$ and for each point $y \in Y$ there is a point $a \in X$ with*

$$g_s(a, y) = \inf_{x \in X} g_s(x, y) \quad \textit{for each} \quad s \in S\,.$$

Then for each subupper limit set-valued map $H : X \to 2^Y$ there is a point $a \in X$ and $b \in H(a)$ such that

$$g_s(a, b) = \inf_{x \in X} g_s(x, b) \quad \textit{for each} \quad S \in S\,.$$

Proof. Let $\{h_n : X \to Y \mid n = 1, 2, \ldots\}$ be a basic family of continuous maps for the subupper limit set-valued map H. According to Infimum Principle for each n there is a point $a_n \in X$ such that

$$g_s(a_n, h_n(a_n)) = \inf_{x \in X} g_s(x, h_n(a_n)) \quad \text{for each} \quad s \in S\,.$$

Compactness implies that there is converging subsequence such that

$$\lim_{k\to\infty} (a_{n_k}, h_{n_k}(a_{n_k})) = (a, b)\,.$$

By definition of limit set-valued map, $b \in H(a)$. Continuity of maps g_s, and $\bar{g}_s(y) := \inf_{x \in X} g_s(x, y)$, $s \in S$ implies that the equality $g_s(a, b) = \inf_{x \in X} g_s(x, b)$ holds for each $a \in S$. □

Theorem 17. *(Gale-Nikaido Theorem). Let $H : \Delta_n \to 2^C$ be an upper semicontinuous map from the n-dimensional standard simplex Δ_n such that $H(x)$ is a non-empty closed and convex subset of a compact convex set $C \subset \mathbf{R}^n$. Suppose further the Walras law in the general sense holds,*

$$\langle x,y\rangle = \sum_{i=1}^{n}(x_i y_i) \geq 0 \quad \textit{for each} \quad x \in \Delta_n \quad \textit{and} \quad y \in H(x).$$

Then there exists $a \in \Delta_n$ and $b \in H(a)$ such that $b_i \geq 0$, for each $i = 1, \dots, n$.

Proof. Applying Infimum Principle for Multivalued Maps to $X = \Delta_n$, $Y = C$, the given set-valued map H, the function g_1 given by $g_1(x, y) = \langle x,y\rangle$ there is a point $(a, b) \in \Delta_n \times C$ such that $b \in H(a)$ and $\langle a,b\rangle = \inf\{\langle x,b\rangle : x \in \Delta_n\}$. By Walras law $\langle a,b\rangle \geq 0$ and in consequence $0 \leq \langle a,b\rangle \leq \langle x,b\rangle$ for each $x \in \Delta_n$. Since $e_i \in \Delta_n$, $0 \leq \langle e_i, b\rangle = b_i$ for each $i = 1, \dots, n$. □

This theorem was instrumental in proving the existence of a competitive equilibrium for excess supply functions for the workability of decentralized economies (in the Walras sense) (cf. [14]).

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