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Regular Closure and Corresponding $\mathscr{P}_\kappa\lambda$ Versions of $\Diamond$ and $\clubsuit$

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We discuss the notion of regular closure for subsets of $\mathscr{P}_\kappa\lambda$ and establish that this yields a sensible strengthening of stationarity. We then introduce corresponding diamond and club principles for $\mathscr{P}_\kappa\lambda$ and establish their consistency relative to ZFC. We show that in this context, diamond implies club as is the case with $\Diamond_\kappa$ and $\clubsuit_\kappa$ for regular κ. When κ is the successor of a regular cardinal, these principles are natural generalisations of established principles on the ordinals. We also introduce several related problems.

1 Introduction

In analysing combinatorial ideas in the context of $\mathscr{P}_\kappa\lambda$, straightforward results do not always transfer easily from the context of ordinals. This situation is apparent when we consider the straighforward proof that $\Diamond_\kappa \Rightarrow \clubsuit_\kappa$ in the context of $\mathscr{P}_\kappa\lambda$ versions of the guessing principles. In this paper, we transfer the standard proof of $\Diamond_\kappa \Rightarrow \clubsuit_\kappa$ to the context of $\mathscr{P}_\kappa\lambda$ by introducing the notion of regular closure and using principles whose guessing corresponds to the associated (stronger) form of stationarity.

Jensen's $\Diamond$ principle was introduced in [4] and has many applications both in combinatorial set theory and more wide y. A related but weaker principle, $\clubsuit$, was introduced by Ostaszevki in [7]; in fact $\clubsuit$ + CH is equivalent to $\Diamond$, as proved by Devlin (see [7]). The $\Diamond$ principle has several established generalisations in the context of $\mathscr{P}_\kappa\lambda$, some of which are discussed below. However, these principles do

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not tie naturally to corresponding ♣ principles. We will see that by strengthening the guessing to r-stationary sets, the theorem $\Diamond_\kappa \Rightarrow ♣_\kappa$ transfers easily to this context.

We first define the $\Diamond_\kappa$ and $♣_\kappa$ principles.

Definition 1.1 $\Diamond_\kappa$ *holds if there is a sequence* $\langle A_\alpha : \alpha < \kappa \rangle$ *such that* $A_\alpha \subseteq \alpha$ *for all* $\alpha < \kappa$ *and for every* $A \subseteq \kappa$ *the set* $\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ *is stationary in* κ.

$♣_\kappa$ *holds if there is a sequence* $\langle A_\alpha : \lim(\alpha) \text{ and } \alpha < \kappa \rangle$ *such that* A_α *is a cofinal subset of* α *for all limit* $\alpha < \kappa$ *and for every cofinal subset* $A \subseteq \kappa$ *the set* $\{\alpha < \kappa : A \cap \alpha \supseteq A_\alpha\}$ *is stationary in* κ.

The following is well-known.

Theorem 1.2 (Ostaszewski) *Suppose* κ *is regular. Then* $\Diamond_\kappa \Rightarrow ♣_\kappa$.

Proof. Let $\{A_\alpha : \alpha < \kappa\}$ witness $\Diamond_\kappa$. We define $\{B_\alpha : \alpha < \kappa \text{ and } \lim(\alpha)\}$ witnessing $♣_\kappa$. If A_α is cofinal in α then let $B_\alpha = A_\alpha$. Otherwise let B_α be any cofinal subset of α.

Given cofinal $X \subseteq \kappa$, let $C_X = \{\alpha < \kappa : X \cap \alpha \text{ is cofinal in } \alpha\}$. Then it is straightforward to show that C_X is club in κ. Consequently, $\{\alpha \in \kappa : X \cap \cap \alpha = A_\alpha\} \cap C_X$ is stationary in κ. Since for all α in this intersection we have $A_\alpha = B_\alpha$, we are done. □

Even within the theory of cardinals, a number of other variations on $\Diamond_\kappa$ have been studied. Jech's $\spadesuit(\kappa, \lambda)$, as defined in [3], was the first generalisation of $\Diamond_\kappa$ to $\mathscr{P}_\kappa\lambda$. A number of other variations have also been defined, notably by Matet in [5] and Džamonja in [2]. The principle defined in Section 3 is a slight variation on Matet's $\bar{\Diamond}_{\kappa,\lambda}$ principles, which is presented below for ease of comparison.

Definition 1.3 $\bar{\Diamond}_{\kappa,\lambda}$ *holds iff there is a set* $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ *such that* $A_x \subseteq \mathscr{P}_{|x}(x)$ *and if* $A \subseteq \mathscr{P}_\kappa\lambda$ *then the set* $\{x : A \cap \mathscr{P}_{|x|}(x) = A_x\}$ *is stationary in* $\mathscr{P}_\kappa\lambda$.

This principle is consistent relative to ZFC if κ is either Mahlo or the successor of a regular cardinal. A $\bar{♣}_{\kappa,\lambda}$ corresponding to $\bar{\Diamond}_{\kappa,\lambda}$ can certainly be defined by insisting A_x is cofinal in $\mathscr{P}_{|x|}(x)$ and requiring guessing only of cofinal $A \subseteq \mathscr{P}_\kappa\lambda$. Although this principle is consistent relative to ZFC, its relationsip to $\bar{\Diamond}_{\kappa,\lambda}$ is unclear. The proof of Theorem 1.2 presented above cannot be transferred because for an arbitrary cofinal $X \subseteq \mathscr{P}_\kappa\lambda$ the set $C_X = \{x \in \mathscr{P}_\kappa\lambda : X \cap \mathscr{P}_{|x|}(x) \text{ is cofinal in } \mathscr{P}_{|x|}(x)\}$ is cofinal in $\mathscr{P}_\kappa\lambda$ but is not closed in general. This is not to say that C_X does not contain a club, which whould be enough for the theorem, but it is not clear whether this is the case.

Another observation that we should make at this point is that when $\kappa = \mu^+$ for regular μ, the principle is unable to distinguish between A, $B \subseteq \mathscr{P}_\kappa\lambda$ for which $A \cap \mathscr{P}_\mu\lambda = B \cap \mathscr{P}_\mu\lambda$. This is simply because for all $A \subseteq \mathscr{P}_\kappa\lambda$ and all $x \in \mathscr{P}_\kappa\lambda$, $A \cap \mathscr{P}_{|x|}(x) \subseteq \mathscr{P}_\mu\lambda$. In this situation, it may be more interesting to require $A \subseteq \mathscr{P}(x)$ instead of $A \subseteq \mathscr{P}_{|x|}(x)$ as in the following variation.

Definition 1.4 $\bar{\Diamond}_{\kappa,\lambda,}$ *holds iff there is a set* $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ *such that* $A_x \subseteq \mathscr{P}(x)$ *and if* $A \subseteq \mathscr{P}_\kappa\lambda$ *then the set* $\{x : A \cap \mathscr{P}(x) = A_x\}$ *is stationary in* $\mathscr{P}_\kappa\lambda$.

The results described in the final section apply equally well to principle like this latter variation. However, for the sake of brevity, we only give the details for the principles derived directly from Matet's $\bar{\Diamond}_{\kappa,\lambda}$. Note that given a set $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ witnessing $\bar{\Diamond}_{\kappa,\lambda,\subset}$, the set $\{A_x \cap \mathscr{P}_{|x}(x) : x \in \mathscr{P}_\kappa\lambda\}$ witnesses $\bar{\Diamond}_{\kappa,\lambda}$.

2 Regular closure

We define versions of clubs and stationary sets in the context of $\mathscr{P}_\kappa\lambda$ based on the notion of regular closure, with the intention that in some situations this may be a more useful combinatorial object than the usual clubs and stationary sets. Note that regular clubs only make sense for $\mathscr{P}_\kappa\lambda$ when κ is the successor of a regular cardinal or is Mahlo. For successor of singulars and for non-Mahlo limit κ, there will be a club subset of $\mathscr{P}_\kappa\lambda$ with no regular sequences.

Definition 2.1 *A sequence* $\langle x_\alpha : \alpha < \mu\rangle$ *is an r-sequence if it is* $\subset$*-increasing and* μ *is an uncountable regular cardinal and* $|\bigcup\{x_\alpha : \alpha < \mu\}| = \mu$.

A set $C \subseteq \mathscr{P}_\kappa\lambda$ *is* r-club *if (i) for any* $x \in \mathscr{P}_\kappa\lambda$, *there is* $y \in C$ *such that* $x \subseteq y$ *and (ii) if* $\langle x_\alpha : \alpha < \mu\rangle$ *is an r-sequence in* C *then* $\bigcup\{x_\alpha : \alpha < \mu\} \in C$.

A set $S \subseteq \mathscr{P}_\kappa\lambda$ *is* r-stationary *in* $\mathscr{P}_\kappa\lambda$ *if for any r-club* $C \subset \mathscr{P}_\kappa\lambda$, $S \cap C \neq \emptyset$.

Note that in the case of $\mathscr{P}_{\kappa^+}\lambda$, where κ is regular, these notions are equivalent to the κ-clubs and κ-stationary sets defined by Dobrinen in [1], for which closure is required for all sequences of length exactly κ. This is simply because all regular sequences in $[\lambda]^\kappa$ are κ-sequences, and vice versa.

We now show that the r-clubs form a normal filter. Consequently, for any regressive functions on $\mathscr{P}_\kappa\lambda$ we can find an r-stationary set for which f is constant, just as we can find a stationary set with this property. We prove this only for Mahlo κ. When κ is the successor of a regular cardinal, the proof is similar; alternatively, the result follows immediately from the corresponding result for κ-closure, as presented in [1].

Definition 2.2 *Let* F *be a filter on* $\mathscr{P}_\kappa\lambda$. *Then* F *is normal iff it satisfies the following:*

(i) κ*-complete: for any* $\{X_\alpha : \alpha < \xi\} \subset F$, *with* $\xi < \kappa$, $\bigcap\{X_\alpha : \alpha < \xi\} \in F$.
(ii) for any $\gamma \in \lambda$, $\{x \in \mathscr{P}_\kappa\lambda : \gamma \in x\} \in F$.
(iii) for any $\langle X_\alpha : \alpha < \lambda\rangle \in {}^\lambda F$, *the diagonal intersection is in* F, *that is* $\triangle\{X : \alpha < \lambda\} := \{x \in \mathscr{P}_\kappa\lambda : x \in \bigcap\{X_\alpha : \alpha \in x\}\} \in F$

Proposition 2.3 *Suppose* κ *is Mahlo. Then the set* $F = \{X \subseteq \mathscr{P}_\kappa\lambda : X$ *contains an r-club*$\}$ *is a normal filter.*

Proof. We first show that given r-clubs C, D, the intersection $C \cap D$ is r-club. (Given two elements C', $D' \in F$ that are not r-clubs, simply work with r-clubs C, D contained in C', D' respectively.) It is trivial to see that $C \cap D$ is r-closed so we simply need to prove cofinality in $\mathscr{P}_\kappa\lambda$. Given $x \in \mathscr{P}_\kappa\lambda$, find sequences $\langle x_\alpha : \alpha < \kappa\rangle$ and $\langle y_\alpha : \alpha < \kappa\rangle$ from C and D respectively with $x \subseteq x_0$ and $x_0 \in \mathscr{P}_{|y_\alpha|}(y_\alpha)$ and $y_\alpha \subseteq x_{\alpha+1}$ for all $\alpha < \kappa$. This can be done because C and D are both cofinal in $\mathscr{P}_\kappa\lambda$ and κ is a limit cardinal. Since κ is Mahlo, there must be some regular limit $\mu < \kappa$ such that $|\bigcup\{y_\alpha : \alpha < \mu\}| = \mu$ since the y_α are increasing in cardinality. Note that $\bigcup\{x_\alpha : \alpha < \mu\} = \bigcup\{y_\alpha : \alpha < \mu\}$ and this is the limit of a regular sequence in both C and D. Hence, it is clear that $\bigcup\{y_\alpha : \alpha < \mu\} \in C \cap D$. Since $x \subseteq \bigcup\{y_\alpha : \alpha < \mu\}$, we are done. By a minor extension of the argument above for closure under pairwise intersections, it follows that F is κ-complete.

It is clear that for all $\gamma \in \lambda$ the set $\{y \in \mathscr{P}_\kappa\lambda : \gamma \in y\} \in F$, since these sets are r-clubs.

Finally, we must observe that F is diagonally closed. That is, for any sequence $\langle X_\xi : \xi < \lambda\rangle$ taken from F, the set $\triangle\{X_\xi : \xi < \lambda\} = \{x \in \mathscr{P}_\kappa\lambda : x \in \in \bigcap\{X_\xi : \xi \in x\}\} \in F$.

We first show that $\triangle\{X_\xi : \xi < \lambda\}$ is cofinal in $\mathscr{P}_\kappa\lambda$. Suppose $x_0 \in \mathscr{P}_\kappa\lambda$. We will find an element $x_\mu \in \triangle\{X_\xi : \xi < \lambda\}$ such that $x_0 \subset x_\mu$. Note that $\{X_\xi : \xi \in x_0\}$ is a set of $< \kappa$-many elements of F and hence the intersection is also in F. Thus we can find $x_1 \in \bigcap(X_\xi : \xi \in x_0\}$ such that $x_0 \in \mathscr{P}_{|x_1|}(x_1)$. Note that we do not necessarily have $x_1 \in \bigcap\{X_\xi : \xi \in x_1\}$ so we are not yet done. We continue inductively, defining $x_{\alpha+1} \in \bigcap\{X_\xi : \xi \in x_\alpha\}$ such that $x_\alpha \in \mathscr{P}_{|x_{\alpha+1}|}(x_{\alpha+1})$ in a similar way. At limit stages, we set $x_\alpha = \bigcup\{x_\beta : \beta < \alpha\}$. Since κ is Mahlo and the x_α are increasing in cardinality, there must be some regular limit stage μ at which $|x_\mu| = \mu$. Note that for all $\alpha < \mu$ the inductive definition implies that for all $\beta \geq \alpha$, $x_\beta \in \bigcap\{X_\xi : \xi \in x_\alpha\}$. Consequently, for all $\alpha < \mu$ the sequence $\langle x_\beta : \alpha < \beta < \mu\rangle$ is an r-sequence in the r-club $\bigcap\{X_\xi : \xi \in x_\alpha\}$ so $x_\mu \in \bigcap\{X_\xi : \xi \in x_\alpha\}$. Hence, for each $\xi \in x_\alpha$, $x_\mu \in X_\xi$. Since for all $\xi \in x_\mu$ there is $\alpha < \mu$ such that $\xi \in x_\alpha$, it follows that $x_\mu \in \bigcap\{X_\xi : \xi \in x_\mu\}$ and hence that $x_\mu \in \triangle\{X_\xi : \xi < \lambda\}$.

We now show that $\triangle\{X_\xi : \xi < \lambda\}$ is r-closed. So suppose $\langle x_\alpha : \alpha < \mu\rangle$ is an r-sequence in $\triangle\{X_\xi : \xi < \lambda\}$ and let $x_\mu = \bigcup\{x_\alpha : \alpha < \mu\}$. We show that $x_\mu \in \triangle\{X_\xi : \xi < \lambda\}$. For all $\alpha < \mu$, the sequence $\langle x_\beta : \alpha \leq \beta < \mu\rangle$ is an r-sequence of $\bigcap\{X_\xi : \xi \in x_\alpha\}$ because for each β in the interval (α, μ), $x_\beta \in \bigcap\{X_\xi : \xi \in x_\beta\} \subseteq \bigcap\{X_\xi : \xi \in x_\alpha\}$. Hence for all $\xi \in x_\alpha$, $x_\mu \in X_\xi$. Since for all $\xi \in x_\mu$, there is $\alpha < \mu$ such that $\xi \in x_\alpha$, it follows that $x_\mu \in \bigcap\{X_\xi : \xi \in x_\mu\}$ and hence that $x_\mu \in \triangle\{X_\xi : \xi < \lambda\}$ as required. □

It is clear that r-stationarity and stationarity are not equivalent. For example, for limit κ, the set $S = \{x \in \mathscr{P}_\kappa\lambda : |x| \text{ is singular}\}$ is stationary but not r-stationary. A more interesting question is whether the principle is equivalent to various other strengthenings of stationarity. We briefly discuss some other stationarity properties below, before moving on to guessing principles in the next section.

One natural generalisation of stationarity corresponds to clubs relativised to a stationary set, as defined below. In the context of cardinals, it is trival to show that r-stationarity and stationarity relativised to the set of regular cardinals are equivalent. However, in the context of $\mathscr{P}_\kappa\lambda$, the situation is less clear.

We use the following notation.

Definition 2.4 *For $X \subset \mathscr{P}_\kappa\lambda$, we say $reg(X) = \{x \in X : |x| \text{ is regular}\}$. Also, we write "reg" for $reg(\mathscr{P}_\kappa\lambda)$.*

Definition 2.5 *Given Mahlo κ we say $C \subseteq \mathscr{P}_\kappa\lambda$ is club^{reg} if there is a club $D \subseteq \mathscr{P}_\kappa\lambda$ such that $C = D \cap reg(\mathscr{P}_\kappa\lambda)$. We say $S \subseteq \mathscr{P}_\kappa\lambda$ is stationary^{reg} if for any club^{reg} $C \subset \mathscr{P}_\kappa\lambda$, $S \cap C \neq \emptyset$.*

Relativising to a stationary set is a useful way of transferring theorems to the $\mathscr{P}_\kappa\lambda$ context but is not effective in the problem discussed in section 3. In [8], Zwicker works with a version of club and stationary set slightly different from the relativised version given above. We subscript with Z here to distinguish the two. Note that in [8], while the intention is to use a stationary coding set, the definition is more general, allowing any arbitrary stationary set. We say $C \subseteq \mathscr{P}_\kappa\lambda$ is club$_Z^{reg}$ in $\mathscr{P}_\kappa\lambda$ if there is a club $D \subseteq \mathscr{P}_\kappa\lambda$ such that $C \cap \operatorname{reg}(\mathscr{P}_\kappa\lambda) = D \cap \operatorname{reg}(\mathscr{P}_\kappa\lambda)$. Also, $S \subseteq \mathscr{P}_\kappa\lambda$ stationary$_Z^{reg}$ if S has non-empty intersection with every club$_Z^{reg}$ set. In fact, the two concepts give the same normal filter and the corresponding definitions of stationarity are equivalent. Note that, assuming κ is Mahlo, any club^{reg} set is also r-club so trivially, r-stationarity implies stationarity^{reg}. It is not clear whether r-stationarity and stationarity^{reg} are equivalent.

In [6], Mignone introduced another form of closure in the interest of studying the combinatorics of $\mathscr{P}_\kappa\lambda$. Here, we say $C \subseteq \mathscr{P}_\kappa\lambda$ is L-closed if for any $\subseteq$-increasing sequence $\langle x_\alpha : \alpha < \delta \rangle$ with $|\delta| \leq \bigcup \{|x_\alpha| : \alpha < \delta\} < \kappa$, which we call a L-sequence, we have $\bigcup \{x_\alpha : \alpha < \delta\} \in C$. Note that $\langle \alpha : \alpha < \omega_1 \rangle$ is an example of a $\subseteq$-increasing sequence that is not an L-sequence. With L-club and and L-stationary defined in the obvious way, the following theorem shows that L-stationarity and stationarity are also not equivalent.

Theorem 2.6 (Mignone)*Suppose κ is huge and λ is any cardinal greater than κ such that there exists a normal ultrafilter over $[\lambda]^\kappa$. Then there is an L-club that does not contain a club.*

Although similar in definition, L-stationarity and r-stationarity appear to be quite distinct. It is trivial that L-stationarity does not imply r-stationarity since the set $\{x \in \mathscr{P}_\kappa\lambda : |x| \text{ is singular}\}$ is L-stationary but not r-stationary.

3 Guessing principles in the context of $\mathscr{P}_\kappa\lambda$

We now define a $\diamond$ principle based on regular stationarity along with a corresponding $\clubsuit$ principle.

Definition 3.1 *$\Diamond^r_{\kappa,\lambda}$ holds iff there is a set $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ such that $A_x \subseteq \mathscr{P}_{x|}(x)$ and if $A \subseteq \mathscr{P}_\kappa\lambda$ then the set $\{x : A \cap \mathscr{P}_{|x|}(x) = A_x\}$ is r-stationary in $\mathscr{P}_\kappa\lambda$.*

$\clubsuit^r_{\kappa,\lambda}$ holds iff there is a set $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ such that A_x is cofinal in $\mathscr{P}_{x|}(x)$ and if A is cofinal in $\mathscr{P}_\kappa\lambda$ then the set $\{x : A \cap \mathscr{P}_{|x|}(x) \supseteq A_x\}$ is r-stationary in $\mathscr{P}_\kappa\lambda$.

Note that $\Diamond^r_{\kappa,\lambda} \Rightarrow \bar{\Diamond}_{\kappa,\lambda}$. We will see below that $\Diamond^r_{\kappa,\lambda} \Rightarrow \clubsuit^r_{\kappa,\lambda}$.

Theorem 3.2 *Suppose M is a countable transitive model of a suitable fragment of ZFC in which $\lambda \geq \kappa$ and either*
(a) κ is Mahlo or
(b) $\kappa = \mu^+$ for some regular μ and $\mu^{<\mu} = \mu$ and $\kappa^{<\kappa} = \kappa$.

Then there is a generic extension $M[G]$ preserving cardinalities, cofinalities and (if appropriate) the Mahlo-ness of κ such that $M[G] \vDash \Diamond^r_{\kappa,\lambda}$.

We proceed by forcing. Let $p \in Q$ iff p is a function with $\mathrm{dom}(p) \in \mathscr{P}_\kappa(\mathscr{P}_\kappa\lambda)$ and $\forall x \in \mathrm{dom}(p)(p(x) \subseteq \mathscr{P}_{|x|}(x))$. For $p, q \in Q$, we say $p \leq q$ iff $q \subseteq p$.

Claim 1. Suppose $\kappa^{<\kappa} = \kappa$. Then Q preserves cardinalities, cofinalities and $\mathscr{P}_\kappa\lambda$ and the fact that κ is Mahlo.

Proof of Claim 1. We show that Q is κ-directed closed and has the κ^+-cc.

First, let $\{p_\alpha : \alpha < \mu\} \subseteq Q$ be a set of pairwise compatible conditions, with $\mu < \kappa$. Then let $p = \bigcup\{p_\alpha : \alpha < \mu\}$. It is clear that $p \in Q$ and for all $\alpha < \mu$, $p \leq p_\alpha$. Thus, Q is κ-directed closed.

That Q has the κ^+-cc follows by a straightforward Δ-system argument. Let $A \subseteq Q$ with $|A| = \kappa^+$ and let $\mathscr{A} = \{\mathrm{dom}(p) : p \in A\}$. Note that since $\kappa^{<\kappa} = \kappa$, it follows that $|\mathscr{A}| = \kappa^+$. If not, there would be κ^+-many elements of A all with the same domain. But this cannot be For suppose $|X|$ is such a domain and $\nu = \sup\{|x| : x \in X\}$. (Note that $\nu < \kappa$.} Then the number of possible conditions is at most $|X|.2^{\nu^{<\nu}} = 2^\nu \leq \kappa^{<\kappa} < \kappa^+$. Note that for the preceding statement, if $\kappa = \mu^+$, we require $\mu^{<\mu} = \mu$ for the first equality and $\kappa^{<\kappa} = \kappa$ for the final inequality. For the case with κ Mahlo, both follow from the assumption that κ is a strong limit.

By the Δ-System Lemma we can find $R \subseteq \mathscr{P}_\kappa\lambda$ with $|R| < \kappa$ and $\mathscr{B} \subseteq \mathscr{A}$ such that $|\mathscr{B}| = \kappa^+$ and for all $X, Y \in \mathscr{B}$, $X \cap Y = R$. Since there are at most κ-many suitable functions that can be defined with domain R, it follows by the pigeonhole principle that there is a set $C \subseteq A$ with $|C| = \kappa^+$ such that for all $p, q \in C$ and all $x \in R$, $p(x) = q(x)$. But it is clear that $p \cup q \in Q$ and $p \cup q \leq p, q$. Thus, A is not an antichain.

□ (Claim 1)

Claim 2. Suppose $M \vDash$ "κ is Mahlo". Let G be a generic of Q. Then $M[G] \vDash \Diamond^r_{\kappa,\lambda}$ holds.

Proof of Claim 2. Given a generic G of Q, for each $x \in \mathscr{P}_\kappa\lambda$, we set $A_x = p(x)$ where p is any element of G with $x \in \operatorname{dom}(G)$. Then $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ is a witness to $\Diamond^r_{\kappa,\lambda}$, as we demonstrate below.

Suppose $p \Vdash$ "$\underset{\sim}{C}$ is club and $\underset{\sim}{A} \subseteq \mathscr{P}_\kappa\lambda$". We find $q \geq p$ such that $q \Vdash$ "$\exists x \in \operatorname{dom}(q)(\underset{\sim}{A} \cap \mathscr{P}_{|x}(x) = q(x)$ and $x \in \underset{\sim}{C})$".

Suppose $p_0 \leq p$ and $x_0 \in \mathscr{P}_\kappa\lambda$ are such that $p_0 \Vdash$ "$x_0 \in \underset{\sim}{C}$". Note that since Q is $<\kappa$-closed we can assume that x_0 is an actual element of $\mathscr{P}_\kappa\lambda$ rather than a name.

Now let $p_0' \leq p_0$ be such that $p_0' \Vdash$ "$\underset{\sim}{A} \cap \mathscr{P}_{|x_0|}(x_0) = A_0$". Note that again by $< \kappa$-closure we can assume A_0 is a set not just a name, since κ is a strong limit so $|\mathscr{P}_{|x_0}(x_0)| < \kappa$.

Now continue inductively as follows. Given p_α, p_α', x_α and A_α, we define $p_{\alpha+1}$ and $x_{\alpha+1}$ so that $p_{\alpha+1}' \leq p_{\alpha+1} \leq p_\alpha'$ and $x_\alpha \in \mathscr{P}_{|x_{\alpha+1}|}(x_{a+1})$ and $p_\alpha \Vdash$ "$x_\alpha \in \underset{\sim}{C}$". We then define $p_{\alpha+1}' \leq p_{\alpha+1}$ such that $p_{\alpha+1}' \vdash$ "$\underset{\sim}{A} \cap \mathscr{P}_{|x_{\alpha+1}|}(x_{\alpha+1}) = A_{\alpha+1}$".

At limit stages α at which $\bigcup\{x_\beta : \beta < \alpha\}$ has singular cardinality, we set $p_\alpha^* = \bigcup\{p_\beta : \beta < \alpha\}$ then find $p_\alpha \geq p_\alpha^*$ and x_α such that $\bigcup\{x_\beta : \beta < \alpha\} \in \mathscr{P}_{|x_\alpha|}(x_\alpha)$ and $p_\alpha \Vdash$ "$x_\alpha \in \underset{\sim}{C}$". As for the successor case, we then define $p_\alpha' \geq p_\alpha$ such that $p' \Vdash$ "$A \cap \mathscr{P}_{|x_\alpha|}(x_\alpha) = A_\alpha$".

We continue the induction until we reach a stage μ at which $\bigcup\{x_\beta : \beta < \alpha\}$ has regular cardinality. This must occur eventually since κ is Mahlo and the x_α are increasing in cardinality. Let $y = \bigcup\{x_\alpha : \alpha < \mu\}$ and let $A_\mu = \bigcup\{A_\alpha : \beta < \mu\}$. We then define q as follows. If there is $\alpha < \mu$ with $x \in \operatorname{dom}(p_\alpha)$ then $q(x) = p_\alpha(x)$. Also, let $q(y) = \mathscr{P}_{|y|}(y) \cap A_\mu$. For all other $z \in \mathscr{P}_\kappa\lambda$, $q(z)$ is undefined. Since $|y|$ is regular and the sequence $\{x_\alpha : \alpha < \mu\}$ is cofinal in $\mathscr{P}_{|y|}(y)$, we have $q \Vdash$ "$\underset{\sim}{A} \cap \mathscr{P}_{|y|}(y) = q(y)$". And since $q \Vdash$ "$\underset{\sim}{C}$ is r-club", we also have $q \Vdash$ "$y \in \underset{\sim}{C}$", since $q \Vdash$ "y is the supremum of a regular sequence in $\underset{\sim}{C}$".

□ (Claim 2)

Claim 3. Suppose $M \models$ "$\kappa = \mu^+$ for some regular cardinal μ". Let G be a generic of Q. Then $M[G] \models \Diamond^r_{\kappa,\lambda}$ holds.

Proof of Claim 3. The proof follows exactly as for Mahlo κ but the x_α all have cardinality μ and we simply require that for all $\beta < \alpha$, $x_\beta \subset x_\alpha$. Note that to obtain p_α' we use the fact that $\mu^{<\mu} = \mu$. The induction stops after μ-many stages.

□ (Claim 3) □

The following is a simple variation of a theorem given in [3] concerning the ♠ principle.

Proposition 3.3 *If κ is a Mahlo cardinal then $\bar{\Diamond}_{\kappa,\lambda}$ holds then there is a family of $2^{|\mathscr{P}_\kappa\lambda}$-many stationary subsets of $\mathscr{P}_\kappa\lambda$ such that the intersection of any two is not stationary.*

By an analogous argument, it can be seen that with $\Diamond^r_{\kappa,\lambda}$ we have a similar theorem, replacing the first instance of "stationary" with "r-stationary". Note that the second instance is still "stationary". The conclusion is apparently stronger since r-stationary sets are in a sense 'larger' since they must intersect not only the clubs but also the r-clubs.

Proposition 3.4 *Suppose κ is a Mahlo cardinal. If $\Diamond^r_{\kappa,\lambda}$ holds then there is a family of $2^{|\mathscr{P}_\kappa\lambda|}$-many r-stationary subsets of $\mathscr{P}_\kappa\lambda$ such that the intersection of any two is not stationary.*

Proof. Let $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ witness $\Diamond^r_{\kappa,\lambda}$. For any $B \subseteq \mathscr{P}_\kappa\lambda$, let $S_B = \{y \in \mathscr{P}_\kappa\lambda : : B \cap \mathscr{P}_{|y|}(y) = A_y\}$. Suppose $B_1, B_2 \subseteq \mathscr{P}_\kappa\lambda$ with $B_1 \neq B_2$. Let $x \in \mathscr{P}_\kappa\lambda$ be such that either $x \in B_1 \setminus B_2$ or $x \in B_2 \setminus B_1$. If $y \in S_{B_1} \cap S_{B_2}$ then $B_1 \cap \mathscr{P}_{|y|}(y) = A_y = B_2 \cap \cap \mathscr{P}_{|y|}(y)$ so $x \notin \mathscr{P}_{|y|}(y)$. Hence $S_{B_1} \cap S_{B_2}$ does not intersect the club $\{y \in \mathscr{P}_\kappa\lambda : x \in \in \mathscr{P}_{|y|}(y)\}$ so is clearly non-stationary. Thus, $\{S_B : B \subseteq \mathscr{P}_\kappa\lambda\}$ is a family of $2^{|\mathscr{P}_\kappa\lambda|}$-many r-stationary subsets of $\mathscr{P}_\kappa\lambda$ each pair of which is non-stationary, as required.

□

The theorem does not hold in the case when κ is the successor of a regular cardinal. In this case, where $\kappa = \mu^+$, say, the set $\{y \in \mathscr{P}_\kappa\lambda : x \in \mathscr{P}_{|y|}(y)\}$ is not a club for any $x \in |\lambda|^\mu$. However, a similar theorem will hold for such a κ if we consider the principle $\Diamond^r_{\kappa,\lambda,\subseteq}$, defined simply by replacing "$A_x \subseteq \mathscr{P}_{|x|}(x)$" with "$A_x \subseteq \mathscr{P}(x)$" in the definition of $\Diamond^r_{\kappa,\lambda}$. (Note also that we can prove the relative consistency of $\Diamond^r_{\kappa,\lambda,\subseteq}$ with a forcing analogous to that given for $\Diamond^r_{\kappa,\lambda}$.) The set $\{y \in \mathscr{P}_\kappa\lambda : x \in \mathscr{P}(y)\}$ is club in $\mathscr{P}_\kappa\lambda$ for all $x \in \mathscr{P}_\kappa\lambda$ so the following theorem for $\Diamond^r_{\kappa,\lambda,\subseteq}$ will proceed as before even for κ the successor of a regular cardinal. Of course, an analogous theorem holds for $\bar{\Diamond}_{\kappa,\lambda,\subseteq}$.

Proposition 3.5 *Suppose κ is either a Mahlo cardinal or the successor of a regular cardinal. If $\Diamond^r_{\kappa,\lambda,\subseteq}$ holds then there is a family of $2^{|\mathscr{P}_\kappa\lambda|}$-many r-stationary subsets of $\mathscr{P}_\kappa\lambda$ such that the intersection of any two is not stationary.*

The following theorem is proved in the same manner as the well-known proof of $\Diamond_\kappa \Rightarrow ♣_\kappa$. Note that we cannot prove that $\bar{\Diamond}_{\kappa,\lambda}$ implies $\bar{♣}_{\kappa,\lambda}$ in this way as the correspondening set C_X (using the notation of the proof below) is not closed under arbitrary unions. Indeed, working with stationary^{reg}, we would have the same problem. Note that it is unclear whether the set C_X *contains* a club or club^{reg} set which would, or course, be sufficient for the theorem.

Theorem 3.6 *Suppose κ is either the successor of a regular cardinal or is Mahlo and that $\lambda \geq \kappa$ Then $\Diamond^r_{\kappa,\lambda}$ implies $♣^r_{\kappa,\lambda}$.*

Proof. Let $\{A_x : x \in \mathscr{P}_\kappa\lambda\}$ witness $\Diamond^r_{\kappa,\lambda}$. We define $\{B_x : x \in \mathscr{P}_\kappa\lambda\}$ witnessing $♣^r_{\kappa,\lambda}$. If A_x is cofinal in $\mathscr{P}_{|x|}(x)$ then let $B_x = A_x$. Otherwise let B_x be any cofinal subset of $\mathscr{P}_{|x|}(x)$.

Given cofinal $X \subseteq \mathscr{P}_\kappa\lambda$, let $C_X = \{x \in \mathscr{P}_\kappa\lambda : X \cap \mathscr{P}_{|x|}(x)$ is cofinal in $\mathscr{P}_{|x|}(x)\}$. We will show that C_X is r-club in $\mathscr{P}_\kappa\lambda$. Consequently, it will follow that

$\{x \in \mathscr{P}\lambda : X \cap \mathscr{P}_{x|}(x) = A_x\} \cap C_X$ is r-stationary in $\mathscr{P}_\kappa\lambda$. Since for all $x \in C_X$ we have $A_x = B_x$, we will be done.

We first show that C_X is cofinal in $\mathscr{P}_\kappa\lambda$. Suppose first that κ is Mahlo. Let $x_0 \in \mathscr{P}\lambda$. We find $y \supseteq x_0$ wıth $y \in C_X$. Since X is cofinal in $\mathscr{P}_\kappa\lambda$, we can find $x_1 \in X$ with $x_0 \subseteq x_1$. Indeed, we can assume without loss of generality that $|x_0| < |x_1|$ since κ is a limit cardinal. We continue inductively, defining $x_{\alpha+1} \in X$ with $x_\alpha \in \mathscr{P}_{|x_{\alpha+1}|}(x_{\alpha+1})$ and taking the union at limit stages until we reach a limit stage μ at which $\bigcup\{x_\alpha : \alpha < \mu\}$ has regular cardinality. Since κ is Mahlo there must be some stage at which the induction stops. Now let $y = \bigcup\{x_\alpha : \alpha < \mu\}$. By the inductive definition, the set $\{x_\alpha : \alpha < \mu\}$ is cofinal in $\mathscr{P}_{|y|}(y)$. Thus $y \in C_X$ and $x_0 \subseteq y$ as required.

If $\kappa = \mu^+$ for some regular μ, the proof that C_X is cofinal is very similar to the one above for Mahlo κ. We require that $|x_\alpha| = \mu$ for all α and continue the induct on for exactly μ-many steps.

We now show that C_X is r-closed. Suppose $\langle x_\alpha : \alpha < \mu\rangle$ is an r-sequence in C_X. Let $y = \bigcup\{x_\alpha : \alpha < \mu\}$. Then by the regularity of μ, for any $z \in \mathscr{P}_{|y|}(y)$, there is $\alpha < \mu$ such that $z \in \mathscr{P}_{x_\alpha|}(x_\alpha)$. Hence there is $u \in C_X \cap \mathscr{P}_{|x_\alpha|}(x_a)$ such that $z \subseteq u$. But clearly $u \in \mathscr{P}_{y|}(y)$. Hence $C_X \cap \mathscr{P}_{y|}(y)$ is clearly cofinal in $\mathscr{P}_{|y}(y)$ so $y \in C_X$ as required.

□

There still remain several natural unanswered problems concerning the relationship between ♣ and ◊ principles in the context of $\mathscr{P}_\kappa\lambda$. Among them are the following.

Question 3.7 *(i) Does* $♣^r_{\kappa,\lambda}$ *imply* $\Diamond^r_{\kappa,\lambda}$*? If not, is there a statement of cardinal arithmetic, "SCA", such that SCA* $+♣^r_{\kappa\lambda}$ *implies* $\Diamond^r_{\kappa,\lambda}$*?*

(ii) Does $\bar{\Diamond}_{\kappa,\lambda}$ *imply* $\bar{♣}_{\kappa,\lambda}$*?*

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