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Measurability of Classes of Lipschitz Manifolds with respect to Borel σ-Algebra of Vietoris Topology

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The measurability of the classes of all k-dimensional Lipschitz manifolds with respects to the Borel σ-algebra of the Vietoris topology on the hyperspace of closed subsets of the d-dimensional Euclidean space is proved. By a k-dimensional Lipschitz manifold we understand a manifold without boundary locally representable by bi-Lipschitz images of closed halfspaces in $\mathbb{R}^k$ or $\mathbb{R}^k$ itself, respectively.

Introduction

The classes of k-dimensional Lipschitz manifolds can be used as a domain of generalized curvature measures (cf. [3]). Further a kinematic formula for this general classes was proved. This enables us to consider the classes of Lipschitz manifolds as an object of interest of stochastic geometry. In this direction, measurability with respect to the usual σ-algebra generated by the Vietoris topology, is needed.

The first section provides an overview, where the Vietoris topology of a hyperspace of all closed subsets of a locally compact, Hausdorff and separable space is introduced, semicontinuity is defined and a relationship between the Vietoris topology and the semicontinuity is briefly described.

Further the class $\mathscr{M}_k$ of k-dimensional Lipschitz manifolds, the class $\mathscr{M}\mathscr{B}_k$ of k-dimensional Lipschitz manifolds with boundary are introduced. Next, the class $\mathscr{M}\mathrm{gr}_{d\ 1}$ of $d - 1$-dimensional strong Lipschitz manifolds without boundary and the class $\mathscr{M}\mathrm{subgr}_d$ of d-dimensional strong Lipschitz manifolds with boundary, defined in [5], are presented. The latest two classes are locally representable as a graph or a subgraph, respectively, of some Lipschitz function defined on $\mathbb{R}^{d-1}$.

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The second section presents a proof of measurability of a general class, whose members are locally characterised by some measurable system (Theorem 10). Finally, measurability of the classes $\mathscr{M}_k$, $\mathscr{M}\mathscr{B}_k$ and the classes $\mathscr{M}\mathrm{gr}_{d-1}$, $\mathscr{M}\mathrm{subgr}_d$ is proved in Theorem 13 and in Theorem 16, respectively, by checking assumptions of Theorem 10.

1. Preliminaries

We operate in $\mathbb{R}^d$, $d \geq 1$, in the whole article. The number $k \in \mathbb{N}$, a dimension of Lipschitz manifolds, is assumed to be between 1 and d. The case of $k = 0$, was already handled in the theory of point processes. Similarly, the case of $d = 1$, $k = 1$, was handled in stochastic geometry (cf. [4]), since Lipschitz 1-manifolds degenerate to countable unions of closed intervals and every closed interval is convex.

Vietoris topology

Let E be a locally compact, Hausdorff and separable space. Then we denote by $\mathscr{F}(E)$ and $\mathscr{G}(E)$ the classes of closed and open subsets of E. We omit the spaces argument if $E \equiv \mathbb{R}^d$, that is we write $\mathscr{F} \equiv \mathscr{F}(\mathbb{R}^d)$.

Further for any $A \subset E$ we define

$$\mathscr{F}_A = \{F: F \in \mathscr{F}(E),\ F \cap A \neq \emptyset\},$$
$$\mathscr{F}^A = \{F: F \in \mathscr{F}(E),\ F \cap A = \emptyset\}.$$

Then the system of classes

$$\mathscr{F}^K_{G_1,\dots,G_n} = \mathscr{F}^K \cap \mathscr{F}_{G_1} \cap \dots \cap \mathscr{F}_{G_n},$$

where $K \subset E$ compact and $G_1, \dots, G_n \in \mathscr{G}(E)$, constitutes a base of Vietoris topology $\tau_{\mathscr{F}(E)}$ on $\mathscr{F}(E)$. It is possible to show, that the Borel σ-algebra of $\tau_{\mathscr{F}(E)}$ is generated by the single classes $\mathscr{F}^K$, $K \subset E$ compact, as well as by the single classes $\mathscr{F}_G$, $G \in \mathscr{G}(E)$.

Theorem 1. *A sequence $\{F_n\}$ converges to F in $\mathscr{F}(E)$ if and only if the two following conditions are satisfied:*

1. *For any $x \in F$ there exists a sequence $\{x_n\}$ with $x_n \in F_n$ such that $x_n \to x$ in E.*
2. *If $\{F_{n_k}\}$ is a subsequence, then every subsequence of points x_{n_k} with $x_{n_k} \in F_{n_k}$ convergent to some $x \in E$ satisfies $x \in F$.*

See [2, 1-2-2]. Note that all the sequences in this article are assumed to be generalized sequences, that is they are defined up to a finite number of members.

Definition 2. *Let Ω be a topological space and Γ a mapping from Ω into $\mathscr{F}(E)$. We say that Γ is upper semicontinuous if for any compact set $K \subset E$ the set*

$\Gamma^{-1}(\mathscr{F}^K)$ *is open in* Ω. *Similarly we say that* Γ *is lower semicontinuous if for any* $G \in \mathscr{G}(E)$ *the set* $\Gamma^{-1}(F_G)$ *is open in* Ω.

It is not difficult to check that a mapping is continuous if and only if it is simultaneously upper and lower semicontinuous.

Proposition 3. *Let* Ω *be a separable topological space and* Γ *a mapping from* Ω *into* $\mathscr{F}(E)$.

1. *The mapping* Γ *is upper semicontinuous if and only if for any* $\omega \in \Omega$, *any sequence* $\{\omega_n\}$ *convergent to* ω *in* Ω *and any sequence* $\{x_n\}$, $x_n \in \Gamma(\omega_n)$, *convergent to some* x *in* E *it holds* $x \in \Gamma(\omega)$.
2. *The mapping* Γ *is lower semicontinuous if and only if for any* $\omega \in \Gamma(\omega)$ *and any sequence* $\{\omega_n\}$ *convergent to* ω *in* Ω *there exists a sequence* $\{x_n\}$ *convergent to* x *in* E *such that* $x_n \in \Gamma(\omega_n)$.

For the proof see [2, 1-2-3; 1-2-4].

Lipschitz Manifolds

Definition 4. *Let* D *be a metric space. We call a function* $\psi: D \to \mathbb{R}$ L-bi-Lipschitz, $L > 0$, *if it satisfies the equation*

$$|x - x'|/L \leq |\psi(x) - \psi(x')| \leq L|x - x'|$$

for every $x, x' \in D$.

Further we call the function ψ bi-Lipchitz *if there exists some* $L > 0$ *such that the function* ψ *is L-bi-Lipschitz.*

Definition 5. *A closed set* $M \subset \mathbb{R}^d$ *is called a* k-dimensional Lipschitz manifold in $\mathbb{R}^d$, $k = 1, \ldots, d$, *if it is locally representable as a bi-Lipschitz image of an open subset of* $\mathbb{R}^k$.

Similarly a k-dimensional Lipschitz manifold in $\mathbb{R}^d$ with boundary *is a set locally representable as a bi-Lipchitz image of a relatively open subset of a closed halfspace in* $\mathbb{R}^k$.

The class of all k-dimensional Lipschitz manifolds (with boundary) will be denoted by $\mathscr{M}_k$ *(*$\mathscr{M}\mathscr{B}_k$*, respectively).*

Further curvature measures for $d - 1$-dimensional manifolds without boundary characterised locally by Lipschitz graphs and for d-dimensional manifolds characterised by Lipschitz subgraphs were constructed in [3].

Definition 6. *If* $d > 1$, *we denote by* $\mathscr{M}\mathrm{gr}_{d-1}$ *and* $\mathscr{M}\mathrm{subgr}_d$ *the classes of closed sets in* $\mathbb{R}^d$ *locally representable by graphs and subgraphs of real Lipschitz functions, respectively, defined on* $\mathbb{R}^{d-1}$. *The members of* $\mathscr{M}\mathrm{gr}_{d-1}$ *and* $\mathscr{M}\mathrm{subgr}_d$ *are called* $d - 1$-dimensional strong Lipschitz manifolds *and* $d - 1$-dimensional strong Lipschitz manifolds with boundary, *respectively.*

The classes of strong Lipschitz manifolds of general dimension were introduced in [5]. Every $d-1$-dimensional strong Lipschitz manifold is also $d-1$-dimensional Lipschitz manifold and that every d-dimensional strong Lipschitz manifold with boundary is also d-dimensional Lipschitz manifold with boundary as well. However the converse does not hold, as is shown in the following counterexample.

Example. Consider a union of broken lines passing subsequently through the points

$$(1,0), (1,1), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{4},0\right), \ldots, \left(\frac{1}{2^j},0\right), \left(\frac{1}{2^j},\frac{1}{2^j}\right), \left(\frac{1}{2^{j+1}},0\right), \ldots \; j \in \mathbb{N}$$

and through the points

$$(-1,0), (-1,1), \left(\frac{-1}{2},0\right), \ldots, \left(\frac{-1}{2^j},0\right), \left(\frac{-1}{2^j},\frac{-1}{2^j}\right), \left(\frac{-1}{2^{j+1}},0\right), \ldots \; j \in \mathbb{N}$$

complemented by the origin (0, 0).

Such a curve cannot be a graph of a function on any neighbourhood of the origin. Hence it cannot be a piece of a $d-1$-dimensional strong Lipschitz manifold. However it can be naturally parameterised by arc to get bi-Lipschitz parameterisation. Therefore it can constitute a local parameterisation of some $d-1$-dimensional Lipschitz manifold.

The curve used in this example can be easily extended to become a d-dimensional Lipschitz manifold with boundary not being any d-dimensional strong Lipschitz manifold with boundary.

2. Measurability

For any open set $G \in \mathcal{G}(\mathbb{R}^d)$ we define the mapping

$$\begin{aligned} \textstyle\bigcap_G : \mathcal{F} &\rightarrow \mathcal{F}(G) \\ F &\mapsto F \cap G. \end{aligned}$$

Lemma 7. *The mapping $\bigcap_G$ is continuous.*

Proof. We will show using Proposition 3 that the mapping $\bigcap_G$ is simultaneously upper and lower semicontinous.

To check the upper semicontinuity choose a sequence $\{F_n\}$ in $\mathcal{F}$ convergent to some $F \in \mathcal{F}$ and a sequence $\{x_n\}$ in G, such that $x_n \to x \in G$ and $x_n \in F_n \cap G$. (If this is not possible, the upper semicontinuity follows trivially.) Then $x \in F$ according to Theorem 1, that is $x \in F \cap G$.

To verify the lower semicontinuity choose some $x \in F \cap G$ and a sequence $\{F_n\}$ $\mathcal{F}$ convergence to F, if both exist. Then, again by Theorem 1, there exists a sequence $\{x_n\}$ convergent to x with $x_n \in F_n$. Since $x \in G$ it holds $x_n \in G$ for almost all $n \in \mathbb{N}$, that is $x_n \in F_n \cap G$ for almost all n. □

Lemma 8. *For any open set $G \subset \mathbb{R}^d$ and any closed subclass $\mathscr{L}$ of $\mathscr{F}$ the system $\bigcap_G(\mathscr{L})$ is closed in $\mathscr{F}(G)$.*

Proof. Choose a sequence $\{F_n\}$ in $\bigcap_G(\mathscr{L})$ convergent to some F in $\mathscr{F}(G)$. Thus there exists a sequence $\{L_n\}$ in $\mathscr{L} \subset \mathscr{F}$ with $L_n \cap G = F_n$. Then there exists a subsequence L_{n_k} convergent to some $L \in \mathscr{L}$ since $\mathscr{F}$ is compact and $\mathscr{L}$ is closed. Due to the continuity of the mapping $\bigcap_G$ (Lemma 7) it holds

$$F_n = \bigcap_G(L_{n_k}) \to \bigcap_G(L) = F. \qquad \square$$

Corollary 9. *For every open set $G \subset \mathbb{R}^d$ and any closed subclass $\mathscr{L}$ of $\mathscr{F}$ the system $\bigcap_G^{-1}(\bigcap_G(\mathscr{L}))$ is a closed subclass of $\mathscr{F}$.*

Proof. Every pre-image of closed set under continuous map is closed by the definition of continuity. The statement thus follows from continuity of the mapping $\bigcap_G$ (Lemma 7). $\square$

Theorem 10. *Let $\mathscr{L}$ be an F_σ subclass of $\mathscr{F}$, and let $\mathscr{M} \subset \mathscr{F}$ be a system of closed sets locally representable by members of the subclass $\mathscr{L}$, that is*

$$\mathscr{M} = \{M \in \mathscr{F};\ \forall x \in M\ \exists G \in \mathscr{G}(\mathbb{R}^d)\colon x \in G \text{ and } M \in \textstyle\bigcap_G^{-1}(\bigcap_G(\mathscr{L}))\}. \tag{1}$$

Then the system $\mathscr{M}$ is measurable in $\mathscr{F}$.

Proof. For the purpose of the proof, let $\mathscr{B}$ denote a countable topological base of $\mathbb{R}^d$ consisting of balls. Given $R > 0$, we denote $\mathfrak{B}_R$ the system of all finite covers $\mathscr{B}_R \subset \mathscr{B}$ of the closed, centred ball $\overline{\mathrm{B}}_R \subset \mathbb{R}^d$. Then the system $\mathfrak{B}_R$ is countable.

Notice, that if B, G are open with $\mathrm{B} \subset G$, then

$$\textstyle\bigcap_G^{-1}(\bigcap_G(\mathscr{L})) \subset \bigcap_{\mathrm{B}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L})).$$

Moreover there are some closed systems $\mathscr{L}_i \subset \mathscr{F}$, $i \in \mathbb{N}$, with $\mathscr{L} = \bigcup \mathscr{L}_i$. Thus the system $\bigcap_G^{-1}(\bigcap_G(\mathscr{L}))$ is F_σ subclass of $\mathscr{F}$, since is satisfies

$$\textstyle\bigcap_G^{-1}(\bigcap_G)(\bigcup \mathscr{L}_i)) = \bigcup \bigcap_G^{-1}(\bigcap_G(\mathscr{L}_i)),$$

where the summands of the right-hand side are close (see Corollary 9).

We wish to show that the following representation holds

$$\mathscr{M} = \bigcap_{R \in \mathbb{N}} \bigcup_{\mathscr{B}_R \in \mathfrak{B}_R} \bigcap_{\mathrm{B} \in \mathscr{B}_R} (\textstyle\bigcap_{\mathrm{B}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))). \tag{2}$$

The measurability of $\mathscr{M}$ would than follow from the measurability of the right-hand side of the above equation.

To show that M is a subclass of the right-hand side of (2) choose some $M \in \mathscr{M}$ and $R \in \mathbb{N}$. The family $\bigcup \{\mathrm{B} \in \mathscr{B};\ M \in \bigcap_{\mathrm{B}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))\}$ covers M due to representation (1) and because $\mathscr{B}$ is topological base. Similarly, there exists a system $\mathscr{B}_{M^C} \subset \mathscr{B}$ with $M^C = \bigcup_{\mathrm{B} \in \mathscr{B}_{M^C}} \mathrm{B}$, $M^C = \mathbb{R}^d \setminus M$, since M^C is open.

Thus the system $\bigcup\{\mathrm{B}\in\mathscr{B};\ M\in\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))\}\bigcup\mathscr{B}_{M^C}$ covers $\overline{\mathrm{B}}_R$ and there exists a finite subcover $\mathscr{B}_R$ such that for every $\mathrm{B}\in\mathscr{B}_R$

1. there exists $G\in\mathscr{G}(\mathbb{R}^d)$ such that $\mathrm{B}\subset G$, that is $M\in\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))$, or
2. $\mathrm{B}\cap M=\emptyset$ which implies $M\in\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))$ as well

Hence we have proved that $M\in\bigcap_{\mathrm{B}\in\mathscr{B}_R}(\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L})))$ and

$$\mathscr{M}_d\subset\bigcap_{R\in\mathbb{N}}\ \bigcup_{\mathscr{B}_R\in\mathfrak{B}_R}\ \bigcap_{\mathrm{B}\in\mathscr{B}_R}(\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))).$$

To prove the converse inclusion consider some member M of the right-hand side of (2) and choose $x\in M$. Then there is some $R\in\mathbb{N}$ such that $x\in\overline{\mathrm{B}}_R$ and some finite cover $\mathscr{B}_R$ of $\overline{\mathrm{B}}_R$ with $M\in\bigcap_{\mathrm{B}\in\mathscr{B}_R}(\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L})))$. This means there is some $\mathrm{B}\in\mathscr{B}_R$ such that $x\in\mathrm{B}$ and $M\in\bigcap_{\overline{\mathrm{B}}}^{-1}(\bigcap_{\mathrm{B}}(\mathscr{L}))$, which gives $M\in\mathscr{M}$. □

Classes of bi-Lipschitz images and Lipschitz graphs

Let R_d denote the group of rotations on $\mathbb{R}^d$ provided with the usual topology. Thus R_d is compact. For any $k=1,\dots,d$ and any $L>0$ we denote

- by $\mathscr{L}_{k,L}\subset\mathscr{F}$ the subclass of image of L-bi-Lipschitz mappings $\psi\colon\mathbb{R}^k\to\mathbb{R}^d$ supplemented by the empty set,
- by $\mathscr{L}\mathscr{B}_{k,L}$ the class of images of L-bi-Lipschitz mappings defined on $\mathbb{R}^k$ or on some closed halfspace of $\mathbb{R}^k$, supplemented by the empty set as well.
- To define $\mathscr{L}\mathrm{gr}_L$, the class of closed sets representable as L-Lipchitz graph, we need to introduce a class Lip-gr$_L$ of L-Lipschitz graphs, that is

$$\begin{aligned}\text{Lip-gr}_L&=\{\text{graph }\psi;\ \psi\colon\mathbb{R}^{d-1}\to\mathbb{R}\text{ is }L\text{-Lipschitz}\},\\ \mathscr{L}\mathrm{gr}_L&=\{\varrho F;\ F\in\text{Lip-gr}_L,\ \varrho\in\mathrm{R}_d\}.\end{aligned}$$

- Similarly by $\mathscr{L}\mathrm{subgr}_L$, we define the class of closed sets representable as L-Lipschitz subgraphs, that is

$$\begin{aligned}\text{Lip-subgr}_L&=\{\text{subgraph }\psi;\ \psi\colon\mathbb{R}^{d-1}\to\mathbb{R}\text{ is }L\text{-Lipschitz}\},\\ \mathscr{L}\mathrm{subgr}_L&=\{\varrho F;\ F\in\text{Lip-subgr}_L,\ \varrho\in\mathrm{R}_d\}.\end{aligned}$$

Lemma 11. *For any $L>0$ and $k=1,\dots,d$ the class $\mathscr{L}_{k,L}$ is closed in $\mathscr{F}$.*

Proof. Consider a sequence $\{L_n\}$ *of nonempty* $L_n\in\mathscr{L}_{k,L}$ convergent to some nonempty closed set $F\in\mathscr{F}$. If $x_0\in F$ then there exists a sequence $\{x_{0,n}\}$, $x_{0,n}\in L_n$ convergent to x_0 according to Theorem 1. Without loss of generality we may assume that $\psi_n^{-1}(x_{0,n})=:a_0$ is independent of $n\in\mathbb{N}$, where ψ_n are L-bi-Lipschitz functions with $L_n=\psi_n(\mathbb{R}^k)$.

We will construct a convergent subsequence $\{\psi_{n_j}\}$ of the sequence $\{\psi_n\}$ by diagonal choice method.

For $j=1$ take some $x_1\in F$ different from x_0 and a sequence of $\{x_{1,n}\}$, $x_{1,n}\in L_n$ convergent to x_1. (Such a point x_1 exists due to the bi-Lipschitz property of the functions ψ_n.) The bounded sequence $\{\psi_n^{-1}(x_{1,n})\}$ has a subsequence $\{\psi_{n_k^1}^{-1}(x_{1,n_k^1})\}$, $k \in \mathbb{N}$, convergent to some point $a_1 \in \mathbb{R}^k$. Hence

$$\lim_{k\to\infty} \psi_{n_k^1}(a_i) = x_i,\ i = 0,1.$$

To finish the first step of our diagonal choice set $\psi_{n_1} := \psi_{n_1^1}$.

Proceeding by induction, for any $j \in \mathbb{N}$ we can take $x_j \in F$ different from x_i, $i = 0, \dots, j-1$, to produce a subsequence $\{\psi_{n_k^j}^{-1}(x_{j,n_k^j})\}$, $k \in \mathbb{N}$, convergent to some $a_j \in \mathbb{R}^k$ out of the bounded sequence $\{\psi_{n_k^{j-1}}^{-1}(x_{j,n_k^{j-1}})\}$, $k \in \mathbb{N}$. We have

$$\lim_{k\to\infty} \psi_{n_k^j}(a_i) = x_i,\ i = 0, \dots, j.$$

To end up the j-th step of the diagonal choice set $\psi_{n_j} := \psi_{n_j^j}$.

Thus we have constructed the sequence $\{\psi_{n_j}\}$ with property, that for every $j \in \mathbb{N}$ it holds

$$\lim_{k\to\infty} \psi_{n_k}(a_j) = x_j.$$

By letting x_j's to exhaust some countable dense subset of F we ensure that the sequence $\{a_j\}$ is dense in $\mathbb{R}^k$. Namely if $a \in \mathbb{R}^k$, denote X_j the set of all cluster points of the bounded sequence $\{\psi_{n_k^j}(a)\}$, $k \in \mathbb{N}$. The intersection $\bigcap_{j\in\mathbb{N}} X_j$ is nonempty, since the sets X_j are nonempty, bounded and form a nested sequence, that is $X_j \supset X_{j+1}$, $j \in \mathbb{N}$.

Next, choose some $x \in \bigcap_{j\in\mathbb{N}} X_j \subset F$ and a subsequence $\{x_{j_k}\}$, $k \in \mathbb{N}$, of $\{x_j\}$ convergent to x. Using bi-Lipschitz property of the functions ψ_n it holds

$$\lim_{k\to\infty} a_{j_k} = \lim_{k\to\infty} \psi_{n_k^{j_k}}^{-1}(x_{j_k}) = a.$$

Hence, the L-bi-Lipschitz mappings ψ_{n_j} converge pointwise on the dense set $\{a_j;\ j \in \mathbb{N}\}$ and due to their bi-Lipschitz they converge uniformly on any compact set to an L-bi-Lipschitz limit ψ defined on $\mathbb{R}^k$, which satisfies the equation $\psi(\mathbb{R}^k) = F$. □

Lemma 12. *For any $L > 0$ and $k = 1, \dots, d$ the class $\mathscr{LB}_{k,L}$ is closed in $\mathscr{F}$.*

Proof. Consider some sequence $\{L_n\}$ of $L_n \in \mathscr{LB}_{k,L}$ convergent to some nonempty set $F \in \mathscr{F}$ and denote by ψ_n the L-bi-Lipschitz functions parameterising L_n with $\psi_n(H_n)$, where $H_n \subset \mathbb{R}^k$ are closed halfspaces or the whole $\mathbb{R}^k$. Further denote ∂H_n the relative boundaries of the halfspaces H_n. We consider $\partial H_n = \emptyset$ if $H_n \equiv \mathbb{R}^k$.

1. Assume first that the sets $H_n \subset \mathbb{R}^k$ are closed halfspaces for almost all $n \in \mathbb{N}$ and that the sequence of the images $\{\psi_n(\partial H_n)\}$ converge to a nonempty set $F' \subset F$.
   Then there is an L-bi-Lipschitz limit $\psi' : \mathbb{R}^{k-1} \to \mathbb{R}^d$ of the mappings ψ_n partialised to ∂H_n according to Lemma 11 if $k > 1$. If $k = 1$, the assertion

is trivial. Moreover for any $x \in F'$ there exists a sequence of $x_n \in \psi_n(\partial H_n)$ convergent to x and we may, without loss of generality, suppose that $\psi_n^{-1}(x_n) = \psi'^{-1}(x) = 0$, that is $0 \in \partial H_n$ for almost all $n \in \mathbb{N}$. Further we may suppose that $H_n = H \subset \mathbb{R}^k$ since rotations preserve distance. The proof in this case then follows using the same method as in Lemma 11, replacing $\mathbb{R}^k$ by H.

2. Assume now, that the sequence $\{H_n\}$ may contain both $\mathbb{R}^k$ and closed halfspaces in $\mathbb{R}^k$ and that limit of the sequence $\{\psi_n(\partial H_n)\}$ is empty set.
Thus we may, without loss of generality, assume that for some sequence of $\{x_n\}, x_n \in L_n$ convergent to some $x \in F$, $\psi_n^{-1}(x_n) = 0$, independently of $n \in \mathbb{N}$. Hence $d(0, \partial H_n) \to \infty$ (we define $d(0, \emptyset) = \infty$), since existence of any subsequence H_{k_n} with $d(0, \partial H_{k'_n}) < K$ would imply, using Lipschitz property, that every cluster point of $\psi_{k_n}(\partial H_{k_n})$ would be nonempty (every $\psi_{k_n}(\partial H_{k_n})$ would have nonempty intersection with the centred ball of diameter LK). That is $H_n \to \mathbb{R}^k$ in $\mathscr{F}(\mathbb{R}^k)$.
Every function ψ_n has an L-Lipschitz extension ψ'_n to $\mathbb{R}^k$ due to the theorem of Kirszbraun and Valentine [1, 2.10.43]. Moreover the sequence of the images $\{\psi'_n(\mathbb{R}^k)\}$ has a cluster point $F' \in \mathscr{F}$ due to compactness of $\mathscr{F}$. A Lipschitz parameterisation ϕ of F' can be constructed in the same way as in the proof of Lemma 11, being a limit of some subsequence $\{\psi'_{l_n}\}$.
It holds $\psi'_{l_n} \equiv \psi_{l_n}$ on H_{l_n}, $n \in \mathbb{N}$, that is $\{\psi_{l_n}\}$ converges to ϕ on every compact set, considering, that every compact is contained in almost all H_{l_n}. The L-bi-Lipschitz property then follows easily.

3. Otherwise, there are at least two different subsequences of $\{\psi_n(\partial H_n)\}$ having different limits F', $F'' \subset F$ and satisfying assumption of 1. or 2. Then the cases 1., 2. imply existence of two different L-bi-Lipschitz parameterisation ψ', ϕ'' of F. But in such a case would F have two different relative boundaries.

Hence only cases 1. or 2. are of possible and it was shown that $F \in \mathscr{LB}_{k,L}$. □

Theorem 13. *The classes $\mathscr{M}_k(\mathscr{MB}_k)$, $k = 1, \ldots, d$, of k-dimensional Lipschitz manifolds in $\mathbb{R}^d$ (with boundary, respectively) are measurable.*

Proof. For any $k = 1, \ldots, d$ the classes $\bigcup_{L\in\mathbb{N}} \mathscr{L}_{k,L}$, $\bigcup_{L\in\mathbb{N}} \mathscr{LB}_{k,L}$ are F_σ in $\mathscr{F}$ according to Lemma 11 and Lemma 12.

The statement then follows from Theorem 10 since representation (1) holds for $\mathscr{M}_k$ and for $\mathscr{MB}_k$ substituting the classes $\bigcup_{L\in\mathbb{N}} \mathscr{L}_{k,L}$, $\bigcup_{L\in\mathbb{N}} \mathscr{LB}_{k,L}$, respectively. □

Lemma 14. *For every $L > 0$ the classes* Lip-gr$_L$ *and* Lip-subgr$_L$ *are closed in* $\mathscr{F}$.

Proof. This statement easy to check since convergence of graphs or subgraphs of L-Lipschitz functions in the Vietoris topology implies locally uniform convergence of these functions. The statement then follows since locally uniform limit of L-Lipschitz functions is L-Lipschitz as well. □

Lemma 15. *For every $L > 0$ the classes $\mathscr{L}\text{gr}_L$ and $\mathscr{L}\text{subgr}_L$ are closed in $\mathscr{F}$.*

Proof. The map

$$\begin{aligned} \Gamma\colon \mathrm{R}_d \times \mathscr{F} &\to F \\ (\varrho, F) &\to \varrho F \end{aligned}$$

is continuous (cg. [4, 1.2.4]) and therefore the set $\Gamma^{-1}(\text{Lip-gr}_L)$ is closed in the product space $\mathrm{R}_d \times \mathscr{F}$. Further the projection $\Pi_{\mathscr{F}} : \mathrm{R}_d \times \mathscr{F} \to \mathscr{F}$ is a closed mapping since R_d is compact. Thus the set $\Pi_{\mathscr{F}}(\Gamma^{-1}(\text{Lip-gr}_L))$ is closed as well.

The closeness of the class $\mathscr{L}\text{gr}_L$ of L-Lipschitz graphs then follows by checking that $\mathscr{L}\text{gr}_L = \Pi_{\mathscr{F}}(\Gamma^{-1}(\text{Lip-gr}_L))$. The closeness of the class $\mathscr{L}\text{subgr}_L$ of L-Lipschitz graphs proceeds in the same way. □

Theorem 16. *The classes $\mathscr{M}\text{gr}_{d-1}$, $\mathscr{M}\text{subgr}_d$ of $d-1$-dimensional strong Lipschitz manifolds and d-dimensional strong Lipschitz manifolds with boundary, respectively, are measurable.*

Proof. The proof proceeds in a similar matter as a proof of Theorem 13, using closeness of the classes $\mathscr{L}\text{gr}_L$ and $\mathscr{L}\text{subgr}_L$ implied by Lemma 15. □

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