Gregory P. Wene

Clifford algebras, matrix algebras and classical groups

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 9. pp. [221]--226.

Persistent URL: http://dml.cz/dmlcz/702132

CLIFFORD ALGEBRAS, MATRIX ALGEBRAS AND CLASSICAL GROUPS

G. P. Wene

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Introduction. There is much discussion in the physics literature concerning associative algebras and their transformation groups. Many of these algebras , the Duffin Kummer algebras, the Dirac algebra, the Majorana algebra, the Clifford algebras and many of their generalizations are simply matrix algebras. There is a very natural association of matrix algebras with the classical semisimple Lie groups. This association was first articulated in the mathematical literature by WEIL (6).

We show, by analyzing the automorphism groups of the Clifford algebras, how to associate with any matrix algebra over either the real numbers or the complex numbers one of the classical Lie groups. We also identify those groups associated, via these techniques, with matrix algebras over the quaternion division algebra H.

2. Basics. Let A denote a finite dimensional algebra over the field R of real numbers or the field C of complex numbers. The algebra A is said to be simple if the only ideals of A are the zero ideal, 0, and A. An algebra A is semisimple if it is an algebra direct sum of simple algebras:

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_n ,$$

where each A_i, $i = 1,2, \ldots, n$, is a simple algebra and the operations of addition and multiplication are defined component-wise. The well-known Wedderburn - Artin Theorem (see JACOBSON (4), page 263) assures us that any semisimple algebra is simply the algebra direct sum of matrix rings. If Δ is a division ring, we will denote by Δ_n the ring of nxn - matrices with entries from Δ under the usual operations.

An <u>automorphism</u> of A is a bijection $\sigma : A \longrightarrow A$ such that
$$(a+b)^{\sigma} = a^{\sigma} + b^{\sigma}$$
and
$$(ab)^{\sigma} = a^{\sigma} b^{\sigma}$$
for all a, b in A. The automorphism σ is <u>inner</u> if $a^{\sigma} = m^{-1}\,a\,m$ for all a in A and some m in A. We have as a corollary to the Noether - Skolem Theorem:

<u>Corollary.</u> Let A be a simple algebra finite dimensional over its center. Then any automorphism of A leaving the center elementwise fixed is inner. For a proof of the Noether - Skolem Theorem and its corollary, see page 199 of HERSTEIN (2).

An <u>involution</u> in A is a mapping $* : A \longrightarrow A$ such that
$$(a^*)^* = a$$
$$(a+b)^* = a^* + b^*$$
$$(ab)^* = b^* a^*$$
for all a, b in A.

<u>3. Examples.</u> In the ring R_n, the mapping $* : R_n \longrightarrow R_n$, defined by $x^* = {}^t x$ (x transpose), is an involution.

If Δ is a division algebra with involution $\overline{}$, then the mapping $* : \Delta_n \longrightarrow \Delta_n$ defined by $x^* = {}^t\bar{x}$ is an involution.

If $A = A_1 \oplus \ldots \oplus A_n$ is an algebra direct sum of simple algebras A_i, $i = 1, \ldots, n$ and $* : A \longrightarrow A$ is an involution then either $*$ maps the summand A_i onto A_i or it interchanges the A_i's in pairs.

In any field K the identity map $x^* = x$ is (trivally) an involution. The real numbers have only the identity map for an involution (or automorphism) (for a proof see page 48 of HEWITT and STROMBERG (3)). The complex numbers have infinitely many involutions (this follows immediately from Exercise 5, page 157 of JACOBSON (5)); most people are familiar with two of these: the identity map and $z^* = \bar{z}$ (z conjugate). The identity map and conjugation are the only continuous involutions (automorphisms) in the usual topology on C.

Let H denote the quaternion division ring. H has a basis over R, $\{1, i, j, k\ (= ij)\}$ such that $i^2 = j^2 = 1$ and $ij = -ji$. The <u>canonical involution</u> in H is the mapping $\overline{}$ defined by: $\bar{1} = 1$, $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$.
This is the involution in H that is used to define the norm, n(x),

of x in H : $n(x) = x\bar{x}$. There are, of course, infinitely many involutions in H.

We will exploit the relation between automorphisms and involutions in matrix algebras. If A is an algebra with involution $*$ we say that the automorphism σ <u>commutes</u> with $*$ if

$$(x^*)^\sigma = (x^\sigma)^*$$

for all x in A. We will denote the group of all automorphisms of A that commute with the involution $*$ by G.

<u>4. Classical Groups And Matrices.</u> A <u>geometry</u> is a triple, $(\Delta^n, M, *)$, where M is an invertible element from Δ_n and $*$ is an involution in Δ. Corresponding to each geometry is a <u>metric</u> or <u>pairing</u> : the metric is the mapping $B : \Delta^n \times \Delta^n \longrightarrow \Delta$ defined by

$$B(x, y) = {}^t x^* \cdot M \cdot y$$

for x, y column vectors in Δ^n and where ${}^t x^*$ is the row vector that is the transpose of X^*.

From a known metric we can find the complete group of transformations of Δ^n with respect to which the metric is a two point invariant. An invertible mapping $\sigma : \Delta^n \longrightarrow \Delta^n$ is called an <u>isometry</u> if

$$B(\sigma(x), \sigma(y)) = B(x, y)$$

for all x, y in Δ^n. The classical groups are isometry groups.

The classical groups are subgroups of $GL(n, \Delta)$, the group of all nxn - matrices over Δ with non-zero determinant. We list these groups, noting that they are defined in terms of involutions in Δ_n.

Let I_n be the n x n - identity matrix and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$SL(n, R)$ ($SL(n, C)$) : The subgroup of $GL(n, R)$ ($GL(n, \Delta)$) of determinant 1.

$O(n, R)$ ($O(n, C)$) : The subgroup of $GL(n, R)$ ($GL(n, C)$) of matrices g satisfying ${}^t g\, g = I_n$.

$SO(n, R)$ ($SO(n, C)$) : The subgroup of $O(n, R)$ ($O(n, C)$) of determinant 1.

$Sp(n, R)$ ($Sp(n, C)$) : The subgroup of $GL(2n, R)$ ($GL(2n, C)$) of matrices g satisfying ${}^t g\, J\, g = J$.

$SU^*(2n)$: The group of matrices in $SL(2n, C)$ which commute with the transformation Ψ of C^{2n} given by

$$(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}) \longrightarrow (\bar{z}_{n+1}, \ldots, \bar{z}_{2n}, -\bar{z}, \ldots, \bar{z}_n).$$

$U(n)$: The subgroup of $GL(n, C)$ of matrices g satisfying ${}^t\bar{g}\, g = I_n$.

5. Classical Groups As Automorphism Groups. Denote by $K^{p,q}$ the Clifford algebra over the field K generated by the elements 1 and e_i, $i = 1, \ldots, p+q$ where

1 is the multiplicative identity,

$$e_i^2 = 1, \qquad 1 \leq i \leq p$$
$$e_i^2 = -1, \qquad p < i \leq p+q$$
$$e_ie_j = -e_je_i, \qquad i \neq j, \quad i, j = 1, \ldots, p+q.$$

If p+q is even, then $K^{p,q}$ is a central simple algebra by CHEVALLEY (1), THEOREM II.2.1. If p+q is odd, then $K^{p,q}$ is either simple or the direct sum of two isomorphic ideals (CHEVALLEY (1), THEOREM II.2.6.). Thus a Clifford algebra is either a matrix algebra or the algebra direct sum of two isomorphic matrix algebras.

RECALL. If A is an algebra with involution $*$, G denotes the group of automorphisms of A that commute with $*$.

THEOREM 5.1. Let $A = K_n \oplus K_n$ with involution

$$(x, y)^* = ({}^ty, {}^tx)$$

for all x, y in K_n. Then G is an algebraic group with connected components G_0 and G_1; G_0 is isomorphic to $PGL(n, K)$, the factor group of $GL(n, K)$ by its center and consists of all automorphisms that leave the summands invariant. The elements of G_1 interchange the summands.

PROOF. We determine G_0. If σ is an element of G_0, then by the Corollary to the Noether-Skolem Theorem

$$(x, y)^\sigma = (M^{-1}\cdot x\cdot M,\ N^{-1}\cdot y\cdot N)$$

for all x, y in K_n and some M, N in K_n. Equating the second components of $((x, y)^\sigma)^*$ and $((x, y)^*)^\sigma$, we get

$${}^tM \cdot {}^tx \cdot {}^tM^{-1} = N^{-1}\cdot {}^tx \cdot N.$$

Hence $(x, y)^\sigma = (M^{-1}\cdot x\cdot M,\ {}^tM\cdot y\cdot {}^tM^{-1})$.

The map $\theta : GL(n, K) \longrightarrow G_0$ is a group homomorphism with kernel the center of $GL(n, K)$. Q.E.D.

THEOREM 5.2. Let $A = K_n$ with involution $*$

$$x^* = {}^tx$$

for all x in K_n. Then G is an algebraic group isomorphic to P O (n, K), the quotient group of O (n, K) by its center.

PROOF. Let σ be an automorphism of A. By the Corollary to the Noether-Skolem Theorem,

$$x^\sigma = M^{-1} \cdot x \cdot M$$

for all x in A and some M in K_n. If σ commutes with $*$, we must have $M^{-1} \cdot {}^t x \cdot M = {}^t M \cdot {}^t x \cdot {}^t M^{-1}$.

Hence $M \cdot {}^t M = I_n$. The matrices in G L (n, K) satisfying this last relation form the group O (n, K). Hence G is isomorphic to P O (n, K). But O (n, K) has two connected components, SO (n, K) and O^-(n, K); the identity component of G, G_o is isomorphic to P S O (n, K), the quotient group of S O (n, K) by its center. Q.E.D.

THEOREM 5.3. Let A = K_{2n} with involution $*$

$$x^* = T^{-1} \cdot {}^t x \cdot T$$

where T is the n x n -diagonal matrix with non-zero entries $\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$. Then G is isomorphic to P S p (n, K), the factor group of S p ((n, K) by its center.

PROOF. Let σ be an element of G. Again,

$$x^\sigma = M^{-1} \cdot x \cdot M .$$

Since σ commutes with $*$, ${}^t M \cdot T \cdot M = T$.

But then M is an element of S p (n, K). Q.E.D.

The proofs of the next two theorems follow in a similar manner and are omitted.

THEOREM 5.4. Let A = C_n with involution $*$,

$$x^* = {}^t\bar{x} \quad \text{(transpose conjugate)}$$

Then G is isomorphic to P U (n), the quotient group of U (n) by its center.

THEOREM 5.5. Let A = $H_n \oplus H_n$ with involution

$$(x, y)^* = ({}^t\bar{y}, {}^t\bar{x})$$

for x, y in H_n where — is the canonical involution in H. Then G has components G_o and G_1, G_o is isomorphic to P G L (n, H) and similar results to those for $K_n \oplus K_n$ follow.

The matrices of determinant 1 form a subgroup of G L (n, H), S L (n, H), which is isomorphic to the group S U^*(2n).

6. Conclusion. We have demonstrated the close connection

between matrix algebras with involutions and the classical Lie groups. We noted that the concept of an algebra with involution is assumed in the definitions of the Lie groups.

REFERENCES

1. C. C. Chevalley, The Algebraic Theory of Spinors. Columbia University Press, New York, 1954.
2. I. N. Herstein. Noncommutative Rings. Carus Mathematical Monograph No. 15, The Mathematical Association of America, Washington, D.C., 1968.
3. E. Hewitt and K. Stromberg. Real And Abstract Analysis. Springer Verlag, Berlin, 1965.
4. N. Jacobson. Structure of Rings. American Mathematical Society Colloquium Publications Vol. XXXVII, American Mathematical Society, Providence, 1956.
5. N. Jacobson. Lectures in Abstract Algebra, Vol. III, Theory of Fields and Galois Theory. Van Nostrand, New York, 1964.
6. A. Weil. " Algebras with involutions and the classical groups" J. Indian Math Soc. 24, 589 -623 (1961).

DIVISION OF MATHEMATICS, COMPUTER SCIENCE AND SYSTEMS DESIGN
THE UNIVERSITY OF TEXAS AT SAN ANTONIO
SAN ANTONIO, TEXAS 78285
U . S . A .