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REMARKS ON CR-MANIFOLDS OF CODIMENSION 2 IN $\mathbb{C}^4$

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Abstract

CR-manifolds of codimension 2 in $\mathbb{C}^4$ carry a geometric structure which is similar to that of real hypersurfaces in complex space. This paper gives a survey on the three local types of these manifolds and their geometric structure. The Cartan's circles that generalise the chains of the hypersurfaces are calculated for the quadratic model manifolds.

1 Introduction

The study of equivalence problem for real hypersurfaces in the complex space with respect to biholomorphic mappings was initiated by Poincaré in 1907 and led to a rather good understanding of the geometry of real hypersurfaces. One of the approaches is based on the construction of a canonical principal fibre bundle with canonical Cartan connection over the hypersurface M (see Cartan [Car32], Tanaka [Tan62], Chern-Moser [CM74], Jacobowitz [Jac77], Webster [Web78]). The structure group of the fibre bundle is the group of isotropic automorphisms of a hermitian quadric Q that is determined by the Levi form of the hypersurface (see below). These groups are well-known groups of fractional linear transformations and they depend only on the signature (p, q) of the Levi form (that is supposed to be nondegenerate).

The quadric Q can be represented as an open subset of the factor space G/P where $G = \operatorname{Aut} Q$ is the group of (fractional linear) automorphisms of Q and $P = \operatorname{Aut}_0 Q$ the subgroup of the automorphisms that preserve the origin. Then the fibre bundle $\pi : \mathcal{G} \to Q$ is nothing but the restriction of the canonical mapping $\operatorname{Aut} Q \to \operatorname{Aut} Q / \operatorname{Aut}_0 Q$ to Q and the Cartan connection coincides with the restriction of the Maurer-Cartan form to $\mathcal{G}$. Jacobowitz [Jac77] used the third order osculation of the hypersurface M by its attached quadric Q to carry the Cartan connection from Q pointwise to M.

It is natural to pose the question about holomorphic equivalence for real submanifolds M of higher codimension in the complex space $\mathbb{C}^N$. As in the case of hypersurfaces

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there is a simple first order invariant: the CR-subspace $T_p^{CR}M$ of the tangent space of M at the point p. This CR-subspace $T_p^{CR}M$ consists of all vectors $v \in T_pM$ such that $Jv \in T_pM$ where J is the operator of multiplication by i in $T_p\mathbb{C}^N$. Usually, one requires that the (complex) dimension n of $T_p^{CR}M$ does not depend on p. Then M is called a CR-submanifold of $\mathbb{C}^N$ and the spaces $T_p^{CR}M$ form a subbundle of the tangent bundle TM.

Any $(2n+k)$-dimensional CR-submanifold of CR-dimension n can be locally reduced to a CR-submanifold of $\mathbb{C}^{n+k}$ by a suitable holomorphic projection (that is diffeomorphic at the manifold itself). By applying the implicit mapping theorem one obtains a local description of M as

$$\operatorname{Im} w_\kappa = f_\kappa(z, \bar{z}, \operatorname{Re} w), \quad \kappa = 1, \dots, k$$

where $z = (z_1, \dots, z_n)$, $w = (w_1 = u_1 + iv_1, \dots, w_k = u_k + iv_k)$ are coordinates in $\mathbb{C}^{n+k}$.

We assume that M is passing through the origin, i.e., $f_\kappa(0) = 0$ for $\kappa = 1, \dots, k$. By applying a linear coordinate change one achieves that $T_0^{CR}M$ coincides with the subspace $w_1 = \cdots = w_k = 0$ and that T_0M coincides with $v_1 = \cdots = v_k = 0$. In these new coordinates the functions f_κ will not have linear parts, i.e., they take the form

$$v_\kappa = \operatorname{Re} b_\kappa(z,z) + h_\kappa(z,\bar{z}) + \operatorname{Re} L_\kappa(z,u) + q_\kappa(u,u) + O(3) \quad (\kappa = 1, \dots, k),$$

where the forms b_κ are quadratic in z , h_κ are hermitian in z, the L_κ are bilinear and the q_κ are quadratic in u, O(3) means terms of order ≥ 3.

By means of the transformation

$$\tilde{w}_\kappa = w_\kappa - ib_\kappa(z,z) - iL_\kappa(z,w) - iq_\kappa(w,w)$$

we eliminate all second order terms, except for the hermitian term (we used that $v_\kappa = O(2)$). Thus, we have found such coordinates that the equation of M is

$$v = h(z, \bar{z}) + O(3), \tag{1}$$

where h is a vector-valued hermitian form. Instead of the notation $h(z,\bar{z})$ we will also write $\langle z, z\rangle$.

On the other hand we will see that no biholomorphic coordinate change can eliminate the hermitian term. In fact, let Φ be a biholomorphic mapping that preserves the origin. Then Φ has the form

$$\begin{aligned} \tilde{z} &= Cz + aw + O(2) \\ \tilde{w} &= Dz + \rho w + O(2). \end{aligned}$$

We see at once that D has to equal 0 because otherwise linear terms would appear in the equation. If we denote by $\tilde{h}(\tilde{z}, \bar{\tilde{z}})$ the k-vector composed by the scalar forms $\tilde{h}_\kappa(\tilde{z}, \bar{\tilde{z}})$ then

$$\tilde{h}(\tilde{z}, \bar{\tilde{z}}) = \rho h(C^{-1}\tilde{z}, \bar{C}^{-1}\bar{\tilde{z}}).$$

Thus, the new forms are nondegenerate linear combinations of the initial forms and they vanish all if and only if the initial forms vanish all. The vector-valued hermitian form $h(z, \bar{z})$ is nothing but the Levi form of M at 0, if $T_0^{CR}M$ is identified with the subspace $w_1 = \cdots = w_k = 0$ and the factor-space $T_0M/T_0^{CR}M$ with the subspace $z_1 = \cdots = z_n = u_1 = \cdots = u_k = 0$.

The CR-manifold M is called Levi nondegenerate at 0 if the components of the Levi form are linearly independent in the space of hermitian forms and if they do not have a common annihilator. It is clear that these properties do not depend on the choice of coordinates. According to a result by Belošapka (see [Bel88]), Levi non-degeneracy implies that point-preserving local holomorphic automorphisms of M are determined by their first and second order derivatives at this point.

2 Non-degenerate CR-quadrics

A CR-manifold Q is called CR-quadric if there are such coordinates that the equation of Q takes the form

$$v = \langle z, z \rangle.$$

The geometric meaning of equation (1) is that any CR-manifold M is osculated by a quadric Q in second order at any fixed point p. In difference to the case of hypersurfaces, where one can achieve third order osculation, this is the maximal possible order. Therefore, one can carry over all second order objects from the quadric that are invariant with respect to its isotropy group to the manifold pointwise. So, it is clear that M and Q are degenerate or nondegenerate at p at the same time.

One of the most significant differences between the hypersurfaces and CR-manifolds of higher codimension is that there are no discrete classifications of the quadrics with respect to linear transformations (by the signature of the hermitian form). This means that the osculating quadrics at arbitrarily close points need not be linearly equivalent and even their automorphism groups may be of different dimension.

One of the few exceptions in higher codimension where one has a discrete classification of the nondegenerate quadrics is the case of codimension 2 in $\mathbb{C}^4$

$$v_1 = \sum_{s,t=1}^{2} h^1_{st} z_s \bar{z}_t$$

$$v_2 = \sum_{s,t=1}^{2} h^2_{st} z_s \bar{z}_t.$$

It turns out that any nondegenerate CR-quadric is linearly equivalent to one out of three standard quadrics that are called hyperbolic, elliptic and parabolic. We will call a point at a CR-manifold M (of codimension 2 in $\mathbb{C}^4$) hyperbolic, elliptic or parabolic in dependence of the osculating quadric at this point.

In order to describe the three types, we consider the "characteristic polynomial" $P(t_1, t_2) = \det(t_1 h^1 + t_2 h^2)$. Though it is not invariant itself, the distribution of the roots will not depend on the choice of the coordinates. A priori, there are four possibilities

1. $P \equiv 0$

2. P has two different real roots (in $\mathbb{RP}^1$)

3. P has one double real root

4. P has two complex mutually conjugate roots.

Let us consider the case when P has at least 2 different real roots. Then there are two linear combination of the two forms that are both degenerate as scalar hermitian forms. This means that

$$\begin{aligned} h^1(z,\bar{z}) &= |c_1^1 z_1 + c_2^1 z_2|^2 \\ h^2(z,\bar{z}) &= |c_1^2 z_1 + c_2^2 z_2|^2. \end{aligned}$$

Since the forms are linearly independent, the matrix $C = (c_m^l)$ is nondegenerate and the vectors $c_1^\kappa z_1 + c_2^\kappa z_2$ can been taken as a new basis in $\mathbb{C}_z^2$. The characteristic polynomial equals now $P = t_1 t_2 \not\equiv 0$. Hence $P \equiv 0$ implies that Q was degenerate. We have obtained the hyperbolic quadric

$$\begin{aligned} h^1(z,\bar{z}) &= |z_1|^2 \\ h^2(z,\bar{z}) &= |z_2|^2. \end{aligned}$$

Now we turn to the case when P has one double root. Without loss of generality, we assume that

$$\begin{aligned} h^1(z,\bar{z}) &= |z_1|^2 \\ h^2(z,\bar{z}) &= h_{12}^2 z_1 \bar{z}_2 + h_{21}^2 z_2 \bar{z}_1 + h_{22}^2 z_2 \bar{z}_2. \end{aligned}$$

Thus, P takes the form

$$P = t_2(h_{22}^2 t_1 + |h_{12}^2|^2 t_2).$$

This polynomial has a double root if and only if $h_{22}^2 = 0$. By setting $2h_{12}^2 z_2 = \tilde{z}$ (and omitting the tilde) we obtain the parabolic quadric

$$\begin{aligned} h^1(z,\bar{z}) &= |z_1|^2 \\ h^2(z,\bar{z}) &= \operatorname{Re} z_1 \bar{z}_2. \end{aligned}$$

It remains to consider the case of two complex conjugate roots. Let $(\lambda_1 : \lambda_2)$, $(\bar{\lambda_1} : \bar{\lambda_2})$ be these two roots in $\mathbb{CP}^1$ and set

$$\begin{aligned} H^1 &= \lambda_1 h^1 + \lambda_2 h^2 \\ H^2 &= \bar{\lambda_1} h^1 + \bar{\lambda_2} h^2. \end{aligned}$$

Then the H^κ are not any more hermitian forms (the coefficients do not satisfy $H_{lm}^\kappa = \bar{H}_{ml}^\kappa$). But the vanishing of the determinant of H^1 means that

$$H^1(z,\bar{z}) = (\bar{z}_1, \bar{z}_2) \begin{pmatrix} \bar{c}_1^1 \\ \bar{c}_2^1 \end{pmatrix} (c_1^2, c_2^2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

With respect to the new coordinates $\tilde{z}_\kappa = c_1^\kappa z_1 + c_2^\kappa z_2$ we have $H^1 = \tilde{z}_1\bar{\tilde{z}}_1$. The hermitian part $\tilde{h}^1$ of H^1 and the skew-hermitian part of H^1 divided by i $\tilde{h}^2$ are real linear combinations of h^1 and h^2 that take the form

$$h^1(z,\bar{z}) = \operatorname{Re} z_1\bar{z}_2$$
$$h^2(z,\bar{z}) = \operatorname{Im} z_1\bar{z}_2$$

(we have omitted the tildes). This is the elliptic quadric.

There is a universal representation of the hyperbolic, elliptic and parabolic quadrics: Consider the spaces $\mathfrak{A}^\delta$ of 2×2-matrices

$$\begin{pmatrix} z_1 & \delta z_2 \\ z_2 & z_1 \end{pmatrix}$$

for $\delta \in \mathbb{R}$. They form commutative subalgebras of the algebra of 2×2-matrices. The equation

$$\operatorname{Im} W = Z\bar{Z}$$

describes CR-quadrics of codimension 2 in the 4 dimensional complex spaces $\mathfrak{A}^\delta_Z \oplus \mathfrak{A}^\delta_W$ with usual complex conjugation. For $\delta = 0$ this is clearly the parabolic quadric. For $\delta > 0$ and $\delta < 0$ we obtain the hyperbolic and elliptic quadrics. To see this one has only to perform the coordinate change

$$\tilde{z}_1 = z_1 + \sqrt{\delta} z_2$$
$$\tilde{z}_2 = z_1 - \sqrt{\delta} z_2.$$

This coordinate change transforms the real part of the algebra $\mathfrak{A}^\delta$ for $\delta > 0$ into the algebra $\mathbb{R} \oplus \mathbb{R}$ represented by diagonal 2×2-matrices and for $\delta < 0$ into $\mathbb{C}$ considered as a real algebra and represented by complex diagonal 2×2-matrices with mutually conjugate entries. The diagonal representations turned out to be most useful in the construction of normal forms (see [Lob88, ES96]).

It is clear that hyperbolicity and ellipticity are stable properties, i.e., there are neighbourhoods of hyperbolic and elliptic points that consist of hyperbolic resp. elliptic points, whereas parabolicity is not stable. The $\mathfrak{A}^\delta$-representation models the behaviour of the CR-manifold in a neighbourhood of a parabolic point.

3 The Lie algebras of infinitesimal automorphisms

The Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \mathfrak{aut}\, Q$ of the quadric Q plays an important role in the construction of Cartan connections because the latter are differential forms with values in $\mathfrak{g}$. As it was shown in [ES94], the infinitesimal automorphisms of the three standard quadrics can be written in the form

$$\sum_{t=1,2} \theta_t \frac{\partial}{\partial z_t} + \omega_t \frac{\partial}{\partial w_t},$$

where θ_t and ω_t are the entries in the matrices

$$\Theta = \begin{pmatrix} \theta_1 & \delta\theta_2 \\ \theta_2 & \theta_1 \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} \omega_1 & \delta\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix},$$

and

$$\Theta = p + XZ + aW + 2i\bar{a}Z^2 2 + rZW, \quad \Omega = q + 2i\bar{p}Z + sW + 2i\bar{a}ZW + rW^2.$$

Here q, r are real and p, a complex elements from $\mathfrak{A}^\delta$. For $\delta \neq 0$, X is a complex element from $\mathfrak{A}^\delta$ and $s = X + \bar{X}$. In the parabolic case

$$X = \begin{pmatrix} X_1 & 0 \\ X_2 & X_1 + t \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} X_1 + \bar{X}_1 & 0 \\ X_2 + \bar{X}_2 & X_1 + \bar{X}_1 + t \end{pmatrix}.$$

Because of the splitting of the hyperbolic quadric into the direct product of two Heisenberg spheres in $\mathbb{C}^2$, the Lie algebra of infinitesimal automorphisms splits into the direct sum of the corresponding Lie algebras for spheres, i.e., into $\mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1)$.

The Lie algebra of infinitesimal automorphisms of the elliptic quadric $\mathfrak{aut}(Q^-)$ turns out to be isomorphic to $\mathfrak{sl}(3,\mathbb{C})$ considered as real Lie algebra. Below, we give the explicit isomorphism. In "diagonal coordinates" the algebra $\mathfrak{aut}(Q^-)$ is spanned by the vector fields

$$\begin{aligned}
\zeta_{-2} &= q\frac{\partial}{\partial w_1} + \bar{q}\frac{\partial}{\partial w_2} \\
\zeta_{-1} &= p_1\frac{\partial}{\partial z_1} + p_2\frac{\partial}{\partial z_2} + 2i\bar{p}_2 z_1\frac{\partial}{\partial w_1} + 2i\bar{p}_1 z_2\frac{\partial}{\partial w_2} \\
\zeta_0 &= X_1 z_1\frac{\partial}{\partial z_1} + X_2 z_2\frac{\partial}{\partial z_2} + (X_1 + \bar{X}_2)w_1\frac{\partial}{\partial w_1} + (X_2 + \bar{X}_1)w_2\frac{\partial}{\partial w_2} \\
\zeta_1 &= \left[a_1 w_1 + 2i\bar{a}_2(z_1)^2\right]\frac{\partial}{\partial z_1} + \left[a_2 w_2 + 2i\bar{a}_1(z_2)^2\right]\frac{\partial}{\partial z_2} + 2i\bar{a}_2 z_1 w_1\frac{\partial}{\partial w_1} + \\
&\quad + 2i\bar{a}_1 z_2 w_2\frac{\partial}{\partial w_2} \\
\zeta_2 &= r z_1 w_1\frac{\partial}{\partial z_1} + \bar{r} z_2 w_2\frac{\partial}{\partial z_2} + r(w_1)^2\frac{\partial}{\partial w_1} + \bar{r}(w_2)^2\frac{\partial}{\partial w_2},
\end{aligned}$$

where $q, p_1, p_2, X_1, X_2, a_1, a_2, r$ are complex numbers. Let E_{ij} be the elementary 3×3-matrices. Then $\mathfrak{sl}(3,\mathbb{C})$ is spanned by E_{ij} with $i \neq j$ and by $h_{ij} = E_{ii} - E_{jj}$. The assignment

$$\begin{aligned}
&E_{12} \mapsto \zeta_{-1} && \text{with} && p_1 = 1, p_2 = 0 \\
&E_{23} \mapsto \zeta_{-1} && \text{with} && p_1 = 0, p_2 = 1 \\
&E_{13} \mapsto \zeta_{-2} && \text{with} && q = -2i \\
&E_{21} \mapsto \zeta_1 && \text{with} && a_1 = 0, a_2 = \frac{-i}{2}
\end{aligned}$$

$E_{32} \mapsto \zeta_1$	with	$a_1 = \frac{-i}{2}$, $a_2 = 0$
$E_{31} \mapsto \zeta_2$	with	$r = \frac{-i}{2}$
$h_{12} \mapsto \zeta_0$	with	$X_1 = 2$, $X_2 = -1$
$h_{23} \mapsto \zeta_0$	with	$X_1 = -1$, $X_2 = 2$

induces the desired isomorphism.

It follows that the algebras in the hyperbolic and elliptic cases are semi-simple.

Now we give a description of the Lie algebra of infinitesimal automorphisms of the parabolic quadric. It is not semi-simple and has dimension 17. This algebra is spanned by the vector fields

$$\begin{aligned}
\zeta_{-2} =& q_1 \frac{\partial}{\partial w_1} + q_2 \frac{\partial}{\partial w_2} \\
\zeta_{-1} =& p_1 \frac{\partial}{\partial z_1} + p_2 \frac{\partial}{\partial z_2} + 2i\bar{p}_1 z_1 \frac{\partial}{\partial w_1} + 2i\left[\bar{p}_1 z_2 + \bar{p}_2 z_1\right] \frac{\partial}{\partial w_2} \\
\zeta_0 =& X_1 z_1 \frac{\partial}{\partial z_1} + \left[(X_1 + t) z_2 + X_2 z_1\right] \frac{\partial}{\partial z_2} + (X_1 + \bar{X}_1) w_1 \frac{\partial}{\partial w_1} + \\
&+ \left[(X_2 + \bar{X}_2) w_1 + (X_1 + \bar{X}_1 + t) w_2\right] \frac{\partial}{\partial w_2} \\
\zeta_1 =& \left[a_1 w_1 + 2i\bar{a}_1 (z_1)^2\right] \frac{\partial}{\partial z_1} + \left[a_2 w_1 + a_1 w_2 + 4i\bar{a}_1 z_1 z_2 + 2i\bar{a}_2 (z_1)^2\right] \cdot \frac{\partial}{\partial z_2} + \\
&+ 2i\bar{a}_1 z_1 w_1 \frac{\partial}{\partial w_1} + 2i\left[\bar{a}_1 (z_2 w_1 + z_1 w_2) + \bar{a}_2 z_1 w_1\right] \frac{\partial}{\partial w_2} \\
\zeta_2 =& r_1 z_1 w_1 \frac{\partial}{\partial z_1} + \left[r_1 (z_2 w_1 + z_1 w_2) + r_2 z_1 w_1\right] \frac{\partial}{\partial z_2} + r_1 (w_1)^2 \frac{\partial}{\partial w_1} + \\
&+ \left[2 r_1 w_1 w_2 + r_2 (w_1)^2\right] \frac{\partial}{\partial w_2},
\end{aligned}$$

where $q_1, q_2, r_1, r_2, t \in \mathbb{R}$, $p_1, p_2, X_1, X_2, a_1, a_2 \in \mathbb{C}$. The vector fields $\zeta_{-2}, \ldots, \zeta_2$ with $q_2 = p_2 = X_2 = t = a_2 = r_2 = 0$ span a subalgebra that is isomorphic to $\mathfrak{su}(2,1)$ and that is the Levi complement to the radical $\mathfrak{r}$. The latter is spanned by the vector fields with $q_1 = p_1 = X_1 = a_1 = r_1 = 0$.

4 Cartan's circles on CR-quadrics

Cartan's circles are canonical families of curves that can be defined at a CR-quadric by using the Maurer-Cartan connection (cf., e.g.,[Sha97], [Slo97]). Let $\mathcal{G} \subset \operatorname{Aut} Q$ be the preimage of the quadric Q under the natural projection $\pi : \operatorname{Aut} Q \to \operatorname{Aut} Q / \operatorname{Aut}_0 Q$. The quadric Q will be identified with a subset of $\operatorname{Aut} Q / \operatorname{Aut}_0 Q$ via the map that assigns to a point p of Q the coset $g \operatorname{Aut}_0 Q$ where g is an automorphism that maps the origin to p, thus $\mathcal{G}$ is the open subset of $\operatorname{Aut} Q$ that consists of the birational automorphisms which are regular at 0.

Consider the orbits of the one-parametric subgroups of $\operatorname{Aut} Q$ that correspond to $\xi \in \mathfrak{g}_{-2}$ with respect to right translations. It is clear, that these orbits are the one-parametric subgroups themselves and their images under left translations. Cartan's circles will be defined as the projections of all these curves with respect to π.

We will take a closer look to the Cartan's circles that pass through the origin and that are contained in the two-dimensional "standard chain" $\Gamma: \ z = 0, \operatorname{Im} w = 0$. In all three cases of non-degenerate quadrics of codimension 2 in $\mathbb{C}^4$ the projections of the one-parametric subgroups themselves are simple lines:

$$z = 0, \ w = qt, \qquad \text{with} \quad q \in \mathbb{R}^2.$$

The other Cartan's circles can be obtained from them by acting with automorphisms that preserve 0 and Γ. Let us start with the hyperbolic case. The automorphisms in question restricted to Γ are

$$\tilde{u}_j = \frac{\rho_j u_j}{1 - r_j u_j}$$

where $u_j = \operatorname{Re} w_j$ are coordinates at Γ and $\rho_j \in \mathbb{R}^*, r_j \in \mathbb{R}$. There are two exceptional Cartan's circles, namely, $u_1 = t, u_2 = 0$ and $u_1 = 0, u_2 = t$ that are invariant with respect to all indicated automorphisms. The other straight lines are contained in one orbit. In this orbit one can find all of the remaining Cartan's circles being hyperbolas:

$$u_j = \frac{\rho_j q_j t}{1 - r_j q_j t},$$

or, as a graph

$$u_1 = \frac{\rho_1 q_1 u_2}{\rho_2 q_2 - (q_1 r_1 - q_2 r_2) u_2}.$$

In the elliptic case we introduce at Γ the complex parameter $\eta = u_1 + i u_2$ with $u_j = \operatorname{Re} w_j$. Then the restriction of the automorphisms are

$$\tilde{\eta} = \frac{c\eta}{1 - r\eta},$$

with $c \in \mathbb{C}^*, r \in \mathbb{C}$. There is only one orbit with respect to the action of these automorphisms. This orbit consists of the initial straight lines and circles (ellipses):

$$\eta = \frac{cqt}{1 - rqt}, \quad \text{with } q \in \mathbb{C}.$$

The equation above is equivalent to

$$\operatorname{Im} \frac{\eta}{q(c + r\eta)} = \operatorname{Im} t = 0,$$

which is, in turn, equivalent to

$$\operatorname{Im} r(u_1^2 + u_2^2) - \operatorname{Im}(\bar{c}\bar{q}) u_1 - \operatorname{Re}(\bar{c}\bar{q}) u_2 = 0.$$

In the parabolic case we have to apply the automorphisms

$$\tilde{u}_1 = \frac{\rho_1 u_1}{1 - r_1 u_1}$$
$$\tilde{u}_2 = \frac{\mu\rho_1 u_1 + \rho_2 u_2 + (\mu\rho_1 r_1 + \rho_2 r_2)(u_1)^2}{(1 - r_1 u_1)^2}$$

to the straight lines in the real plane. There is one exceptional line that forms an orbit $u_1 = 0, u_2 = t$. All the other Cartan's circles are contained in a second orbit. They have the form

$$\tilde{u}_1 = \frac{\rho_1 q_1 t}{1 - r_1 q_1 t}$$
$$\tilde{u}_2 = \frac{(\mu\rho_1 q_1 + \rho_2 q_2)t + (\mu\rho_1 r_1 + \rho_2 r_2)(q_1)_2 t^2}{(1 - r_1 q_1 t)^2}.$$

By eliminating the parameter t we obtain

$$u_2 = \alpha(u_1)^2 + \beta u_1$$

with

$$\alpha = \frac{\mu\rho_1(q_1 + r_1) + \rho_2(q_2 + r_2)}{(q_1)^2}, \qquad \beta = \frac{\mu\rho_1 q_1 + \rho_2 q_2}{\rho_1 q_1}.$$

Thus, in this case the Cartan's circles are straight lines and parabolas.

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