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PROBLEMS WHICH LEAD TO A GENERALIZATION OF THE CONCEPT OF AN ORDINARY NONLINEAR DIFFERENTIAL EQUATION

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This lecture is concerned with general theorems on the continuous dependence of a solution on a parameter, which theorems are closely related to the averaging principle. The main feature of these theorems lies in the assumption that the primitive with respect to the time-variable of the right hand side of the nonlinear differential equation depends continuously on a parameter. This assumption is obviously weaker than the usual assumption that the right hand side itself depends continuously on a parameter. The concept of a generalized differential equation is introduced in order that the examined class of differential equations be closed with respect to the above continuous dependence assumption. The averaging principle is contained in the theorems on the continuous dependence on a parameter. This approach avoids the transformation of the equation to a canonical form and remains valid for differential equations in Banach spaces. Hence some applications to boundary value problems in partial differential equations are made. In this paper several unpublished results are included; such results are denoted by an asterisk.

1 The finite dimensional case

Let us start by applying the averaging principle in the simplest case of a weakly nonlinear system in two dimensions

$$\text{(1)} \qquad \begin{aligned} \frac{dx_1}{dt} &= x_2 + \varepsilon \tilde{f}_1(x_1, x_2, t), \\ \frac{dx_2}{dt} &= -x_1 + \varepsilon \tilde{f}_2(x_1, x_2, t), \\ \tilde{f}_i(x_1, x_2, t + 2\pi) &= \tilde{f}_i(x_1, x_2, t) \quad (i = 1, 2), \end{aligned}$$

ε being a small parameter. By means of the substitution

$$x_1 = u_1 \cos t + u_2 \sin t, \quad x_2 = -u_1 \sin t + u_2 \cos t$$

one obtains

$$\text{(2)} \qquad \begin{aligned} \frac{du_1}{dt} &= \varepsilon f_1(u_1, u_2, t), \\ \frac{du_2}{dt} &= \varepsilon f_2(u_1, u_2, t), \\ f_i(u_1, u_2, t + 2\pi) &= f_i(u_1, u_2, t) \quad (i = 1, 2). \end{aligned}$$

The averaging principle consists in replacing system (2) by

$$\frac{du_1}{dt} = \varepsilon f_{01}(u_1, u_2), \qquad (3)$$

$$\frac{du_2}{dt} = \varepsilon f_{02}(u_1, u_2),$$

for ε small enough. Here

$$f_{0i}(u_1, u_2) = \frac{1}{2\pi} \int_0^{2\pi} f_i(u_1, u_2, t)\, dt \quad (i = 1, 2). \qquad (4)$$

This replacement was justified by a series of results. E. g. it is known that the existence of an exponentially stable solution of (3) implies the existence of an exponentially stable solution of (2) and (1) for ε small and that the existence of an exponentially stable integral variety of (3) in the autonomous case implies the existence of an integral variety of (2) and (1) (see [1], [2]).

Let $u, f, F, \ldots$ denote vectors from an n-dimensional vector space R_n, let $\| \|$ denote a norm in this space. An interesting motivation for the averaging principle is given by a theorem due to I. I. Gihman [3].

Theorem 1. (Gihman.) *Let the right hand sides of*

$$\frac{du}{d\tau} = f(u, \tau, \varepsilon), \quad \varepsilon > 0, \qquad (5)$$

$$\frac{du}{d\tau} = f(u, \tau, 0) \qquad (6)$$

fulfil the following conditions

i) $f(u, \tau, \varepsilon)$ *is defined for* $u \in G \subset R_n$, G *open*, $0 \leqq \tau \leqq T$, $0 \leqq \varepsilon \leqq \varepsilon_0$;

ii) $f(u, \tau, \varepsilon)$ *is continuous in* (u, τ) *for* $0 \leqq \varepsilon \leqq \varepsilon_0$;

iii) $\|f(u, \tau, \varepsilon)\| \leqq K$;

iv) $\|f(u_2, \tau, \varepsilon) - f(u_1, \tau, \varepsilon)\| \leqq K\|u_2 - u_1\|$;

v) $\int_0^\tau f(u, \sigma, \varepsilon)\, d\sigma \to \int_0^\tau f(u, \sigma, 0)\, d\sigma$ *with* $\varepsilon \to 0$.

Denote by $u(\tau, \varepsilon)$, $0 \leqq \tau \leqq T$, $0 \leqq \varepsilon \leqq \varepsilon_0$ *the solution of* (5) *or* (6), *fulfilling the initial condition* $u(0, \varepsilon) = \tilde{u}$. *Then* $u(\tau, \varepsilon) \to u(\tau, 0)$ *uniformly with* $\varepsilon \to 0$, $0 \leqq \leqq \tau \leqq T$.

Denote by $u^*(t, \varepsilon) = (u_1^*(t, \varepsilon), u_2^*(t, \varepsilon))$ the solution of (2) with $u^*(0, \varepsilon) = \tilde{u}$, denote by $u_0^*(t, \varepsilon)$ the solution of (3) with $u_0^*(0, \varepsilon) = \tilde{u}$. It follows from Theorem 1 that

$$\max_{0 \leqq t \leqq T/\varepsilon} \|u^*(t, \varepsilon) - u_0^*(t, \varepsilon)\| \to 0 \quad \text{with} \quad \varepsilon \to 0$$

(under obvious assumptions concerning $f_1, f_2, \tau = \varepsilon t$).

Conditions iii) and iv) of Theorem have been weakened by several authors (see [4], [5], [6]). They may be replaced by the weaker conditions

$$\left\| \int_{\tau_1}^{\tau_2} f(u, \sigma, \varepsilon) \, d\sigma \right\| \leqq \omega_1(|\tau_2 - \tau_1|), \tag{7}$$

$$\left\| \int_{\tau_1}^{\tau_2} f(u_2, \sigma, \varepsilon) \, d\sigma - \int_{\tau_1}^{\tau_2} f(u_1, \sigma, \varepsilon) \, d\sigma \right\| \leqq \|u_2 - u_1\| \, \omega_2(|\tau_2 - \tau_1|). \tag{8}$$

Here ω_1 and ω_2 are nondecreasing and the series

$$\sum_{j=1}^{\infty} 2^j \omega_1(2^{-j}) \, \omega_2(2^{-j}) \tag{9}$$

converges. We may write $\omega_1(\eta) = K\eta^\alpha$, $\omega_2(\eta) = K\eta^\beta$, $K > 0$, $\alpha > 0$, $\beta > 0$ and in this case (9) converges if $\alpha + \beta > 1$. If the solutions of the limit equation (6) are uniquely determined by their initial values, then again the solutions of (5) converge to the solutions of (6) with $\varepsilon \to 0$ (see [6]). The above conditions are fulfilled e.g. in the case

$$\frac{du}{d\tau} = u\varepsilon^{\tilde{\beta}-1} \cos(\tau/\varepsilon) + \varepsilon^{\tilde{\alpha}-1} \sin(\tau/\varepsilon), \tag{10}$$

$$\frac{du}{d\tau} = 0, \tag{11}$$

where $\tilde{\alpha} > 0$, $\tilde{\beta} > \frac{1}{2}$, $\tilde{\alpha} + \tilde{\beta} > 1$. If $\tilde{\alpha} < 1$, $\tilde{\beta} < 1$, then neither the bound of the right hand side of (10) nor the Lipschitz constant may be chosen independently of ε.

Conditions (7), (8) have the following interesting property. Introduce

$$F(u, \tau, \varepsilon) = \int_0^\tau f(u, \sigma, \varepsilon) \, d\sigma .$$

Conditions (7), (8) may be rewritten in terms of F as

$$\|F(u, \tau_2, \varepsilon) - F(u, \tau_1, \varepsilon)\| \leqq \omega_1(|\tau_2 - \tau_1|), \tag{12}$$

$$\|F(u_2, \tau_2, \varepsilon) - F(u_2, \tau_1, \varepsilon) - F(u_1, \tau_2, \varepsilon) + F(u_1, \tau_1, \varepsilon)\| \leqq \leqq \|u_2 - u_1\| \, \omega_2(|\tau_2 - \tau_1|). \tag{13}$$

Also condition v) of Theorem 1 may be rewritten as

$$F(u, \tau, \varepsilon) \to F(u, \tau, 0) \quad \text{with} \quad \varepsilon \to 0 . \tag{14}$$

This means that all conditions which are essential for the convergence of the solutions may be written in terms of F. f is only needed to define solutions of a differential equation. In order to introduce a generalized concept of a differential equation observe that $u(\tau)$ is a solution of $du/d\tau = f(u, \tau)$ if $u(\tau_2) - u(\tau_1)$ is approximately equal to the sum on the right hand side in

$$(15)\qquad u(\tau_2) - u(\tau_1) = \int_{\tau_1}^{\tau_2} f(u(\sigma), \sigma)\,\mathrm{d}\sigma = \sum_{i=1}^{k} \int_{\sigma_{i-1}}^{\sigma_i} f(u(\sigma), \sigma)\,\mathrm{d}\sigma \doteq$$
$$\doteq \sum_{i=1}^{k} [F(u(\zeta_i), \sigma_i) - F(u(\zeta_i), \sigma_{i-1})]\,,$$

the subdivision $\tau_1 = \sigma_0 < \zeta_1 < \sigma_1 < \ldots < \sigma_{k-1} < \zeta_k < \sigma_k = \tau_2$ being fine. This fact motivates the following

Definition. *Let $F(u, \tau)$ be a function of the independent variables u, τ and let $u(\tau)$ be a function of the independent variable τ (no continuity or smoothness assumptions being made). $u(\tau)$ is a solution of the generalized differential equation*

$$(16)\qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = D_\tau F(u, \tau)$$

if $\sum_{i=1}^{k} [F(u(\zeta_i), \sigma_i) - F(u(\zeta_i), \sigma_{i-1})]$ is arbitrarily close to $u(\tau_2) - u(\tau_1)$ for a sufficiently fine subdivision $\tau_1 = \sigma_0 < \zeta_1 < \sigma_1 < \ldots < \sigma_{k-1} < \zeta_k < \sigma_k = \tau_2$.

The essence of this generalization is in the replacement of $\int_{\tau_1}^{\tau_2} f(u(\sigma), \sigma)\,\mathrm{d}\sigma$ by a certain integral of the Stieltjes type. What is meant by a "fine subdivision" is not described here. If this concept is interpreted in the usual way, the limit of the sums $\sum_{i=1}^{k} [F(u(\zeta_i), \sigma_i) - F(u(\zeta_i), \sigma_{i-1})]$ has the properties of the Riemann integral, another interpretation implies that this limit has the properties of the Perron integral (see [6]).

The following theorem clears up the relation of the classical differential equation to the generalized one.

Theorem 2. *If $f(u, \tau)$ fulfils the Carathéodory conditions and if $F(u, \tau) = \int_0^\tau f(u, \sigma)\,\mathrm{d}\sigma$, then every solution of generalized equation (16) is simultaneously a solution of the classical equation*

$$(17)\qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = f(u, \tau)$$

and conversely, every solution of (17) is at the same time a solution of (16) (see [6]).

Under the conditions of Theorem 2, equations (17) and (16) are equivalent. Therefore there arises the problem of finding other conditions on F, which would make it possible to prove the existence theorem for equation (16) and to examine the uniqueness and the continuous dependence on a parameter. It turns out that inequalities (12) and (13) are such appropriate conditions and that a complete theory may be established for equation (16) with F fulfilling (12), (13). Of course, ω_1 and ω_2 are assumed to be nondecreasing and series (9) has to converge.

***Theorem 3.** *If F fulfils* (12) *and* (13) *and if* $u(\tau)$ *is a solution of* (16), *then*

$$\|u(\tau_2) - u(\tau_1)\| \leqq K\,\omega_1(|\tau_2 - \tau_1|) \quad \textit{for} \quad |\tau_2 - \tau_1| \leqq 1$$

with $K > 0$ *dependent on* ω_1, ω_2 *only.*

Theorem 3 plays a similar part as the assertion that every solution of a classical equation with a continuous right hand side fulfils a Lipschitz condition.

Theorem 4. *If F fulfils* (12) *and* (13) *and if* $\tilde{u}$ *and* τ_0 *are given, then there exists a solution* $u(\tau)$ *of* (16) *on an interval containing* τ_0, $u(\tau_0) = \tilde{u}$.

This existence theorem has a local character. Prolongation of solutions is analogous to the case of classical equations.

Theorem 5. *Let* $F(u, \tau, \varepsilon)$ *fulfil* (12), (13), ε *being fixed*, $0 \leqq \varepsilon \leqq \varepsilon_0$ *and let* (14) *be fulfilled. Denote by* $u(\tau, \varepsilon)$ *the solution of*

$$\frac{du}{d\tau} = D_\tau F(u, \tau, \varepsilon) \tag{18}$$

satisfying the initial condition $u(0, \varepsilon) = \tilde{u}$. *Let* $u(\tau, 0)$ *be defined for* $0 \leqq \tau \leqq T$ *and uniquely determined by the initial condition. Then* $u(\tau, \varepsilon) \to u(\tau, 0)$ *uniformly with* $\varepsilon \to 0$, $0 \leqq \tau \leqq T$.

Theorem 5 contains a general formulation of the averaging principle. The set of functions F, which fulfil (12), (13) (ω_1 and ω_2 being fixed) is closed with respect to limiting process (14).

In comparison with the papers already published, Theorem 3 is new. Theorem 3 makes it possible to avoid the concept of the "regular solution", which is found in the earlier formulations of the existence theorem and of the theorem on the continuous dependence on a parameter. Thus Theorems 4 and 5 are simplified in comparison with results contained in published papers (see [6], [7]).

Another problem of importance is the uniqueness of solutions. A simple example shows that the solutions of (16) need not be unique if F fulfils (12) and (13) (see [8]). But when $u(\tau) =$ const. happens to be a solution of (16), then this solution is unique (see [9]).*) If F fulfils (12), (13) and

$$\begin{aligned} \|F(u_2 + v, \tau_2) - F(u_2 + v, \tau_1) - F(u_1 + v, \tau_2) + F(u_1 + v, \tau_1) - \\ - F(u_2, \tau_2) + F(u_2, \tau_1) + F(u_1, \tau_2) - F(u_1, \tau_1)\| \leqq \\ \leqq \|u_2 - u_1\| \cdot \|v\| \cdot \omega_2(|\tau_2 - \tau_1|), \end{aligned} \tag{19}$$

*) The concept of the "regular solution" may be omitted as above.

then all solutions of (16) are unique. Adding (19) to (12) and (13) we strengthen the requirements for the dependence of F on u. This corresponds to the fact that in the case of classical differential equation (17) stronger conditions for the dependence of f on u are needed in the uniqueness theorem than in the existence theorem. It may be shown that new results about classical differential equations are also contained in the above uniqueness results.

In the above conditions (12), (13) it is essential that series (9) converges. Let us put the question, whether or not it is possible to prove a modified Theorem 5 assuming that the right hand side of (18) converges with $\varepsilon \to 0$ (i.e. (14) takes place) and (12) and (13) are fulfilled, $\omega_1(\eta)$ and $\omega_2(\eta)$ being such nondecreasing functions that series (9) diverges. An answer to this question, which may be considered as definite, was obtained by Jiří Jarník. If $\omega_2(\eta)$ is not near to linear function (e.g. if $\omega_2(\eta) \geqq \eta^\alpha$, $0 < \alpha < 1$), then the modified Theorem 5 cannot hold, if series (9) diverges. If $\omega_2(\eta) = c\eta$ or if $\omega_2(\eta)$ is near to a linear function, then the condition that series (9) converges may be weakened (see [10], [11], [12]).

By means of methods based on the above theory, estimates for the distance of solutions of two different differential equations may be obtained. This was performed in a paper by Z. Vorel. In general it may be said that the new approach leads to better results than the classical one if the right hand sides of the differential equations contain terms rapidly oscillating in τ (see [13]).

A special group is formed by cases for which the convergence of solutions of a sequence of differential equations may be proved in spite of the fact that series (9) necessarily diverges. Here, of course, the equation for the limit of the solutions is to be derived in a more complicated manner than in the above theory. In particular systems of a special form, which correspond to differential equations of the second order were examined. Equations of such type describe e.g. the motion of the Kapica's pendulum, i.e. a pendulum hung in a rapidly oscillating point. Even in these rather complicated cases it is possible not only to obtain convergence results but to prove the existence of a stable periodic solution for $\varepsilon \neq 0$. Results in this direction are due to Jiří Jarník and are being prepared for publication (see [14]).

Conditions (12), (13) are not the only ones, from which the basic theorems (i.e. existence, continuous dependence on a parameter and – under some additional assumptions – uniqueness) for equation (16) may be deduced. The basic theorems were derived from

$$(20) \qquad \|F(u, \tau_2) - F(u, \tau_1)\| \leqq |h(\tau_2) - h(\tau_1)|\,,$$

$$(21) \qquad \|F(u_2, \tau_2) - F(u_2, \tau_1) - F(u_1, \tau_2) + F(u_1, \tau_1)\| \leqq \leqq \omega(\|u_2 - u_1\|)\, |h(\tau_2) - h(\tau_1)|\,,$$

where $h(\tau)$ is nondecreasing and continuous from the left and $\omega(\eta) \to 0$ with $\eta \to 0$. Under these conditions the solutions of (16) are functions of bounded variation and, in general, may be continued to the right only. Again, the problems which are connected to the continuous dependence on a parameter, are very interesting, but it is not possible to go into the details here.

2 The infinite dimensional case

Theorem 1 has an advantage that in its proof the assumption that u is a vector from a finite dimensional space was not used. It is well known that the method of successive approximations converges for equation

$$\frac{du}{d\tau} = f(u, \tau), \tag{22}$$

where u is taken from an open subset G of a Banach space B, $0 \leqq \tau \leqq T$, f has its values in B, is continuous and fulfils a Lipschitz condition with respect to u. Here the more general case of the generalized equation

$$\frac{du}{d\tau} = D_\tau F(u, \tau) \tag{23}$$

will be treated. It is supposed that F has its values in B and

$$\|F(u, \tau_2) - F(u, \tau_1)\| \leqq K|\tau_2 - \tau_1|, \tag{24}$$

$$\|F(u_2, \tau_2) - F(u_2, \tau_1) - F(u_1, \tau_2) + F(u_1, \tau_1)\| \leqq \leqq K\|u_2 - u_1\| \, . \, |\tau_2 - \tau_1| \tag{25}$$

for $u, u_1, u_2 \in G$, $\tau_1, \tau_2 \in \langle 0, T \rangle$.

$u(\tau)$ is defined to be a solution of (23) if for every $\delta > 0$

$$\|u(\tau_2) - u(\tau_1) - \sum_{i=1}^{k} [F(u(\zeta_i), \sigma_i) - F(u(\zeta_i), \sigma_{i-1})]\| < \delta \tag{26}$$

whenever the subdivision $\tau_1 = \sigma_0 < \zeta_1 < \sigma_1 < \ldots < \sigma_{k-1} < \zeta_k < \sigma_k = \tau_2$ is sufficiently fine (cf. Definition in section 1). Again, the method of successive approximations converges and in this way existence and uniqueness of solutions is obtained. The results, which will be obtained for the differential equations in Banach spaces, will be applied to some boundary value problems in partial differential equations. In these applications the generalized equation (23) proves more convenient than the classical equation (22). Also some methods used in the proofs are closely connected to the concept of the generalized equation. The following theorem is an extension of Gihman's theorem to generalized equations in Banach spaces.

***Theorem 6.** *Let $F(u, \tau, \varepsilon)$ fulfil* (24), (25), *ε being fixed, $0 \leqq \varepsilon \leqq \varepsilon_0$, K independent of ε and let*

$$F(u, \tau, \varepsilon) \to F(u, \tau, 0) \text{ uniformly with } \varepsilon \to 0\,. \tag{27}$$

Denote by $u(\tau, \varepsilon)$, $0 \leqq \varepsilon \leqq \varepsilon_0$, $0 \leqq \tau \leqq T$ *the solution of*

$$\frac{du}{d\tau} = D_\tau F(u, \tau, \varepsilon)\,, \quad \varepsilon > 0 \tag{28}$$

or of

$$\frac{du}{d\tau} = D_\tau F(u, \tau, 0) \tag{29}$$

satisfying the initial condition $u(0, \varepsilon) = \tilde{u}$, $0 \leqq \varepsilon \leqq \varepsilon_0$. *Then*

$$u(\tau, \varepsilon) \to u(\tau, 0) \text{ uniformly with } \varepsilon \to 0\,.$$

Theorem 6 justifies to some extent the replacement of equation (28), ε being small, by equation (29), as the solutions of both equations behave similarly. However, it does not follow from Theorem 6 that there exists an exponentially stable solution of equation (28), if there exists an exponentially stable solution of equation (29). (A solution $w(\tau)$ defined for $\tau \geqq 0$ is called exponentially stable, if there exist $k > 0$, $\beta > 0$ such that $\|u(\tau) - v(\tau)\| \leqq k\|u(\tau_0) - v(\tau_0)\|\, e^{-\beta(\tau-\tau_0)}$, $\tau \geqq \tau_0 \geqq 0$, for each couple of solutions $u(\tau)$, $v(\tau)$ from some neighbourhood of $w(\tau)$.) But if the assumptions of Theorem 6 are strengthened, the assertion may be strengthened also and then more detailed conclusions about the properties of equation (28) for ε small may be drawn from the properties of the limit equation (29).

***Theorem 7.** *Assume that* $F(u, \tau, \varepsilon)$ *fulfils* (24), (25) *and* (27) *as in Theorem 6 and that, in addition,*

$$\begin{aligned} \|F(u_2 + v, \tau_2, \varepsilon) - F(u_2 + v, \tau_1, \varepsilon) - F(u_1 + v, \tau_2, \varepsilon) + F(u_1 + v, \tau_1, \varepsilon) - \\ - F(u_2, \tau_2, \varepsilon) + F(u_2, \tau_1, \varepsilon) + F(u_1, \tau_2, \varepsilon) - F(u_1, \tau_1, \varepsilon)\| \leqq \\ \leqq K\|u_2 - u_1\| \,.\, \|v\| \,.\, |\tau_2 - \tau_1|\,, \quad 0 \leqq \varepsilon \leqq \varepsilon_0\,, \end{aligned} \tag{30}$$

$$\begin{aligned} \|F(u_2, \tau, \varepsilon) - F(u_1, \tau, \varepsilon) - F(u_2, \tau, 0) + F(u_1, \tau, 0)\| \leqq \\ \leqq \|u_2 - u_1\|\, h(\varepsilon)\,, \end{aligned} \tag{31}$$

$$\begin{aligned} \|F(u_2 + v, \tau, \varepsilon) - F(u_1 + v, \tau, \varepsilon) - F(u_2, \tau, \varepsilon) + F(u_1, \tau, \varepsilon) - \\ - F(u_2 + v, \tau, 0) + F(u_1 + v, \tau, 0) + F(u_2, \tau, 0) - F(u_1, \tau, 0)\| \leqq \\ \leqq \|u_2 - u_1\| \,.\, \|v\| \,.\, h(\varepsilon) \end{aligned} \tag{32}$$

with $h(\varepsilon) \to 0$ *with* $\varepsilon \to 0$. *Denote by* $u(\tau, \varepsilon)$, $v(\tau, \varepsilon)$, $\tau \in \langle 0, T\rangle$, $0 \leqq \varepsilon \leqq \varepsilon_0$ *solutions of* (28) (*or* (29) *for* $\varepsilon = 0$) *satisfying the initial conditions* $u(0, \varepsilon) = \tilde{u}$, $v(0, \varepsilon) = \tilde{v}$. *Then*

$$\|u(\tau, \varepsilon) - v(\tau, \varepsilon) - u(\tau, 0) + v(\tau, 0)\| \leqq \|\tilde{u} - \tilde{v}\|\, \tilde{h}(\varepsilon)\,, \tag{33}$$

$\tau \in \langle 0, T\rangle$, $\tilde{h}(\varepsilon) \to 0$ *with* $\varepsilon \to 0$.

Let us mention that (33) may be replaced by a stronger inequality, which is needed for some purposes, but we shall not go into such details. If $\|u(\tilde{\tau}, 0) - v(\tilde{\tau}, 0)\| \leqq \alpha \|\tilde{u} - \tilde{v}\|$ for some $\tilde{\tau}$ and α, then it follows from (33) that $\|u(\tilde{\tau}, \varepsilon) - v(\tilde{\tau}, \varepsilon)\| \leqq [\alpha + \tilde{h}(\varepsilon)] \cdot \|\tilde{u} - \tilde{v}\|$. Hence it may be concluded that there exists an exponentially stable solution of (28) for $\varepsilon > 0$ small enough, if there exists an exponentially stable solution of the limit equation (29).

In order to ilustrate the contents of Theorem 7 consider the well known equation

$$\frac{d^2x}{dt^2} + x = \varrho_1 \sin \omega_1 t + \varrho_2 \sin \omega_2 t + \varepsilon(1 - x^2) \frac{dx}{dt} . \tag{34}$$

As is well known, there exists an exponentially stable almost periodic solution of (34), if $Q = 4 - 2[\varrho_1^2/(1 - \omega_1^2)^2 + \varrho_2^2/(1 - \omega_2^2)^2] < 0$ and there exists a stable variety filled up by solutions of (34), if $Q > 0$ (a one-parametrical system of periodic solutions, if $\varrho_1 = \varrho_2 = 0$). These facts may be proved from a single point of view by means of Theorem 7.

Theorems 6 and 7 are not bounded to spaces of a finite dimension. Therefore, several examples will be indicated in order to demonstrate the possibility of applying these theorems to boundary value problems in partial differential equations. Consider

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \varepsilon f(x, t, u, u_x, u_t) , \tag{35}$$

$u(0, t) = u(1, t) = 0$. The solution is desired for $0 \leqq x \leqq 1$, $t \geqq 0$. The point of departure is the expression for the solution of (35) for $\varepsilon = 0$, which may be found in the form

$$u(x, t) = S(x + t) + R(x - t) , \tag{36}$$

S and R being functions of a single variable, or in the form

$$u(x, t) = \sum_{k=1}^{\infty} (a_k \cos \pi kt + b_k \sin \pi kt) \sin \pi kx . \tag{37}$$

Starting from (36) we may show that the solution u of (35) with $\varepsilon > 0$ may be represented in the form $u(x, t) = S(x + t, \varepsilon t) + R(x - t, \varepsilon t)$, S and R being functions of two variables. Equation (35), $\varepsilon > 0$ is equivalent to a certain partial differential equation for the function S. This equation may be examined as an ordinary differential equation in Banach space and it turns out that under very general assumptions concerning f the conditions for applying Theorems 6 (see [21]) and 7 are fulfilled. The main difficulty lies in examining the properties of the limit equation.

The limit equation was examined for

$$f = (1 - u^2)\, u_t + \sin \pi x \sin \pi t . \tag{38}$$

It turns out that in this case there exists a periodic solution of problem (35) with period 2, which is exponentially stable in the large.

If f is defined by one of the following formulas

$$f = (1 - u^2)\, u_t\,, \tag{39}$$

$$f = (1 - u_x^2)\, u_t\,, \tag{40}$$

$$f = (1 - u_t^2)\, u_t\,, \tag{41}$$

a complicated situation arises, since the corresponding limit equations have a rich set of solutions independent of τ under which there are even unstable ones. Therefore it is difficult to conclude anything about the properties of problem (35) from the properties of the limit equation. Only in the case (41) can some boundedness results be obtained. If f is defined by (39) or (40), it can be proved that the derivatives u_t and u_x of the solution u of (35) may assume great values even if the initial conditions are smooth and if ε is small (see [21]).

However, in the cases (38) to (41) f is very special. On the other hand, the simple expression (36) of the solution of (35) for $\varepsilon = 0$ makes it possible to obtain rather detailed information in these cases. Unlike the finite dimensional case it seems that the examination of limit equations derived from problems for partial differential equations will always be difficult. But for a certain type of a nonlinear function f it is possible to examine the properties of this case under qualitative assumptions concerning f.

Consider problem (35), where f is defined by

$$f = -\, g(x, u)\, u_t + h(x, t)\,, \tag{42}$$

g and $\partial^2 g/\partial u^2$ being continuous,

$$g(x, u) \geqq g_0 > 0\,, \quad h(x, t + 2) = h(x, t)\,, \quad \int_0^1 h^2(x, t)\, \mathrm{d}x \leqq c < \infty\,.$$

In this case the solution of problem (35) for $\varepsilon = 0$ was desired in the form of (37), a_k and b_k being unknown functions of τ. This leads to a certain ordinary differential equation in a Banach space, the elements of which are sequences $\{a_k, b_k\}$. It is possible to prove that there exists an exponentially stable periodic solution of (35) for $\varepsilon > 0$ small, f being defined by (42). The solution found under the above conditions is a generalized one: u is continuous, u_t is square integrable and equation (35) is fulfilled in the sense of the theory of distributions.

In a quite analogous manner the problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = \varepsilon[-\, g(x, u)\, u_t + h(x, t)]\,, \tag{43}$$

$$u(0, t) = u_{xx}(0, t) = u(1, t) = u_{xx}(1, t) = 0$$

may be examined, the assumptions concerning g and h remaining unchanged. There exists an exponentially stable periodic solution of (43) for $\varepsilon > 0$ small.

These two cases, which were treated in the same manner, indicate that it is possible to proceed further.

Consider

$$\frac{\partial^2 u}{\partial t^2} + L_x u = \varepsilon[- g(x, u)\, u_t + h(x, t)]\,, \tag{44}$$

L_x is a linear differential operator in the variable $x = (x_1, x_2, \ldots, x_n)$ with coefficients defined in an open subset X of the Euclidian space E_n and there are given some boundary conditions. g fulfils the same conditions as in (42), h is uniformly almost periodic in τ. There exists a complete orthonormal system of eigenfunctions φ_k of

$$L_x \varphi = \lambda \varphi \tag{45}$$

with the given boundary conditions, the corresponding eigenvalues λ_k being positive, $k = 1, 2, 3, \ldots$, $|\varphi_k(x)| \leqq K$, $k = 1, 2, 3, \ldots$, $x \in X$, X is bounded.

The eigenvalues λ_k satisfy

$$\frac{\lambda_{k+1}}{\lambda_k} \geqq 1 + \frac{\delta}{k}, \quad \delta > 3 \tag{46}$$

for k great enough. Under these conditions it is possible to prove that there exists an almost periodic exponentially stable solution of (44) for $\varepsilon > 0$ small enough.

However, condition (46) is too restrictive. (It is not fulfilled, if $k^{-2}\lambda_k \to c < \infty$ for $k \to \infty$, which takes place e. g. in case of second order differential operators in one dimension.) This is caused by the fact that, unlike the preceding two special cases, the general case with almost periodic solutions is treated here. There arises the problem, whether or not the above conditions may be modified in such a way as to obtain a satisfactory theory in case (46) is not fulfilled.

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