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NONLINEAR NONCOERCIVE BOUNDARY VALUE PROBLEMS

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1. Introduction

The main purpose of this lecture is to formulate some results and open problems concerning the solvability of nonlinear equations. The results are not presented in full generality and only the ideas are presented. For more general results see the references. But, of course, the list of references is not complete.

Let Ω be a bounded domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$ sufficiently smooth if $N > 1$. Let $g: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ be a continuous function. We are concerned with the solvability of the Dirichlet problem

$$(1) \qquad \begin{cases} -\Delta u(x) - g(u(x)) = f(x) & \text{in } \Omega \\ \qquad\qquad u(x) = 0 & \text{on } \partial\Omega \,. \end{cases}$$

We shall discuss the existence of a weak solution of (1) under the assumption that there exist limits

$$\lim_{\xi\to\infty} \frac{g(\xi)}{\xi} = \mu \,, \qquad \lim_{\xi\to-\infty} \frac{g(\xi)}{\xi} = \nu \,,$$

and we shall consider various configurations of $\mu, \nu \in [-\infty, \infty]$.

2. The case $\mu = \nu = \lambda \in \mathbb{R}^1$

For simplicity suppose that

$$(2) \qquad g(\xi) = \lambda\xi - \psi(\xi) \,, \quad \xi \in \mathbb{R}^1 \,,$$

where $\psi : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ is a bounded and continuous function.

2.1. Regular case

Suppose that the problem

$$(3) \qquad \begin{cases} -\Delta u(x) - \lambda u(x) = 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

has only trivial weak solutions.

Then (according to the Schauder fixed point theorem) the problem (1) with g given by (2) is weakly solvable for arbitrary $f \in L_2(\Omega)$. This is the so-called regular (or coercive) case since the operator

$$(4) \qquad L : u \longmapsto -\Delta u - \lambda u$$

is invertible in the Sobolev space $W_0^{1,2}(\Omega)$. For references on results of such type see e.g. [3].

2.2. Resonance case

2.2.1. Consider the problem (1) (with g given by (2)) under the assumption that the operator L defined by (4) is noninvertible, i.e. the problem (3) has nontrivial weak solutions. Such a case of (1) was considered first in [15]. It is included in the theory of the solvability of nonlinear operator equations of the type

$$(5) \qquad Lu = Su ,$$

where L is linear noninvertible and Fredholm operator between Banach spaces X and Z, and $S: X \longrightarrow Z$ is a nonlinear completely continuous mapping. It is possible to obtain the solvability of the nonlinear equation (5) using the Schauder fixed point theorem (see e.g. [4]); the method of fixed point theorems was used in the special case of differential operators in many papers e.g.

by the authors: J.Cronin, D.G. De Figueiredo, P.Hess, J.Nečas, L. Nirenberg, V.P.Portnov, M.Schechter, S.A.Williams, ... It is also possible to use Mawhin's coincidence degree theory (for the exposition, numerous applications and further references see [13]) or a certain variational approach (see e.g. [11], [12]) .

The results mentioned above, applied to (1) , yield the following theorem.

2.2.2. Suppose that we have, for an arbitrary nontrivial solution w of (3),

$$(6) \qquad \limsup_{\xi \to \infty} \psi(\xi) \int_\Omega w^+(x)dx - \liminf_{\xi \to -\infty} \psi(\xi) \int_\Omega w^-(x)dx < \int_\Omega f(x)\, w(x)\, dx < \liminf_{\xi \to -\infty} \psi(\xi) \int_\Omega w^+(x)dx - \limsup_{\xi \to \infty} \psi(\xi) \int_\Omega w^-(x)dx \ ,$$

where w^+ and w^- are the positive and negative parts of the function w , respectively.

Then the problem (1) is weakly solvable.

The existence of weak solutions of (1) follows also in the case when in (6) the reverse inequalities hold.

2.2.3. If

$$\psi(\infty) < \psi(\xi) < \psi(-\infty) \ , \ \xi \in \mathbb{R}^1$$

($\psi(\infty) = \lim_{\xi\to\infty} \psi(\xi)$, $\psi(-\infty) = \lim_{\xi\to-\infty} \psi(\xi)$) , then (6) is a necessary and sufficient condition for the weak solvability of (1).

2.3. Vanishing nonlinearities

2.3.1. The set of the right hand sides f satisfying (6) may be empty e.g. in the case

(7) $$\psi(-\infty) = \psi(\infty) = 0 .$$

Then, 2.2.2 gives no existence result.

The idea how to prove the existence of weak solutions of (1) with g given by (2) where ψ satisfies (7) is based on the so--called method of truncated equations: we change the function outside a sufficiently large interval $(-A, A)$ such that for the function $\tilde{\psi}$ obtained, condition (6) gives sense. The main part of the proof of existence is to show that an arbitrary weak solution u of (1) with g given by

$$g(\xi) = \lambda\xi - \tilde{\psi}(\xi) , \ \xi \in \mathbb{R}^1$$

satisfies

(8) $$\|u\|_C < A .$$

Thus we obtain the following results under the assumption that ψ is odd.

2.3.2 (see [9]). Consider (1) with $N = 1$. Let $a > 0$ and let

(9) $$\lim_{\xi\to\infty} \xi^2 \min_{\tau\in[a,\xi]} \psi(\tau) = \infty$$

and

(10) $$\int_\Omega f(x)\, w(x)\, dx = 0$$

for an arbitrary weak solution of (3).

Then (1) has at least one weak solution.

2.3.3 (see [10]). Consider (1) with $N > 1$. Let $a > 0$ and let

(11) $$\lim_{\xi\to\infty} \xi \min_{\tau\in[a,\xi]} \psi(\tau) = \infty \quad .$$

Suppose $f \in L_\infty(\Omega) \cap C(\Omega)$ and (10) .

Then (1) has at least one weak solution.

2.3.4. In [14] the result from 2.3.3 is extended under the assumption that

(12) $$\liminf_{\xi\to\infty} \xi \min_{\tau\in[a,\xi]} \psi(\tau) > 0$$

(instead of (11)) and that an arbitrary weak solution of (3) has the so-called "unique continuation property". For further generalization see [11] .

2.3.5. OPEN PROBLEM. To prove the apriori estimate (8) in the case of (1) with $N > 1$ it is necessary to estimate

(13) $$\sup \int_{\Omega_\varepsilon(w)} |w(x)|\, dx \leqq \text{const.} \quad \varepsilon^{1+\rho} \quad ,$$

where the supremum is taken over all the solutions of (3) with $\|w\|_C = 1$ and where

$$\Omega_\varepsilon(w) = \{ x \in \Omega \; ; \; 0 < |w(x)| < \varepsilon \} \quad .$$

Obviously, (13) holds with $\rho = 0$; thus we obtain the sufficient condition (11) in 2.3.3 . If $N = 1$ then (13) holds with $\rho = 1$ and hence we obtain the sufficient condition (9) in 2.3.2 . If (13) were true with some $\rho > 0$, it would be possible to replace (11) by

$$\lim_{\xi\to\infty} \xi^{1+\rho} \min_{\tau\in[a,\xi]} \psi(\tau) = \infty \quad .$$

Probably, a proof of (13) with $\rho > 0$ may be based on the investigation of the nodal lines of the solutions of (3) with using a version of the maximum principle. Unfortunately, we are not aware of any correct result from this field.

2.3.6. OPEN PROBLEM. If the condition (12) is not satisfied (e.g. if ψ has a compact support) nothing is known about the existence of solutions of (1) in the resonance case.

2.4. Expansive nonlinearities

Using the same method as in 2.3 we can investigate also the weak solvability of (1) with g given by (2) in the case that there exists none of the limits $\psi(\infty)$, $\psi(-\infty)$.

2.4.1. A bounded odd continuous and nontrivial function ψ : $\mathbb{R}^1 \longrightarrow \mathbb{R}^1$ is said to be expansive if for each p with

$$0 \leqq p < \sup_{\xi \in \mathbb{R}^1} \psi(\xi)$$

there exist sequences $0 < a_k < b_k$ with

$$\lim_{k \to \infty} b_k a_k^{-1} = \infty$$

such that

$$\lim_{k \to \infty} \min_{\xi \in [a_k, b_k]} \psi(\xi) > p .$$

A typical example of an expansive function which has none of the limits $\psi(\infty)$, $\psi(-\infty)$ is $\psi(\xi) = \sin \xi^{1-\varepsilon}$ with $1 > \varepsilon > 0$.

2.4.2 (see [9], [10]). Considering (1) with g given by (2) with ψ expansive and $f \in L_\infty(\Omega) \cap C(\Omega)$ if $N > 1$ we see that

(14) $$\left| \int_{\Omega} f(x)\, w(x)\, dx \right| < \sup_{\xi \in \mathbb{R}^1} \psi(\xi) \int_{\Omega} |w(x)|\, dx$$

for an arbitrary nontrivial solution w of (3) is a necessary and sufficient condition for the weak solvability of (1) .

2.4.3. OPEN PROBLEM. It seems that nontrivial conditions for the weak solvability of (1) with g given by (2) where $\psi(\xi) = \sin \xi$ (or an analogous periodic function) in the resonance case are so far unknown.

3. Jumping nonlinearities

3.1. $\mu, \nu \in \mathbb{R}^1,\ \mu \neq \nu$

3.1.1. Consider (1) with $N = 1$ and $\Omega = (0, \pi)$. Suppose that

(16) $$g(\xi) = \mu \xi^+ - \nu \xi^- - \psi(\xi)$$

where $\psi : \mathbb{R}^1 \to \mathbb{R}^1$ is bounded and continuous.

3.1.2 (see [8],[2],[6]). Let the assumptions from 3.1.1 be satisfied. Put

$$\mathcal{M} = \{ (\mu, \nu) \in \mathbb{R}^2 ;\ \mu < 1,\ \nu < 1 \} \cup \bigcup_{k=0}^{\infty} \{ (\mu, \nu) \in \mathbb{R}^2;\ \mu^{1/2} > k+1,\ \omega_k(\mu^{1/2}) < \nu^{1/2} < \gamma_{k+1}(\mu^{1/2}) \}$$
$$\cup \bigcup_{k=1}^{\infty} \{ (\mu, \nu) \in \mathbb{R}^2;\ \mu^{1/2} > k,\ \gamma_k(\mu^{1/2}) < \nu^{1/2} < \vartheta_k(\mu^{1/2}) \} ,$$

where

$$\vartheta_k(\tau) = \begin{cases} \dfrac{(k+1)\tau}{\tau - k}, & \tau \in (k, 2k+1] \\ \dfrac{k\tau}{\tau - (k+1)}, & \tau \in (2k+1, \infty) \end{cases},$$

$$\omega_k(\tau) = \begin{cases} \dfrac{k\tau}{\tau - (k+1)}, & \tau \in (k+1, 2k+1] \\ \dfrac{(k+1)\tau}{\tau - k}, & \tau \in (2k+1, \infty) \end{cases},$$

$$\gamma_k(\tau) = \frac{k\tau}{\tau - k}, \quad \tau \in (k, \infty).$$

(i) If $(\mu, \nu) \in \mathcal{M}$, (1) has at least one weak solution for each f.

(ii) If $(\mu, \nu) \notin \overline{\mathcal{M}}$, there exists an $f \in C^\infty$ for which (1) has no weak solution.

3.1.3. OPEN PROBLEMS. If $N > 1$ then for the weak solvability of (1) with g satisfying (16) we can characterize the parameters (μ, ν) only in the following cases:

a) (μ, ν) are close to the diagonal $\{(\lambda, \lambda) \ ; \lambda \in \mathbb{R}^1\}$ (see [2], [6]);

b) $\mu < \lambda_1, \nu < \lambda_1$ (λ_1 is the first eigenvalue of $-\Delta$) - this case is included in the theory of pseudomonotone operators;

c) μ, ν are close to λ_1 (see [1], [5] and the papers of A.Ambrosetti - G.Mancini , M.S.Berger, E.N.Dancer, E.Podolak,...);

d) μ, ν are close to a simple eigenvalue of $-\Delta$ (see [2], [6], and the papers of A.Ambrosetti - G.Mancini , E.Podolak,...) .

For a better description of the solvability of (1) with respect to the values μ, ν it would be important to solve the following problems:

(i) Find all $(\mu, \nu) \in \mathbb{R}^2$ for which

$$\begin{cases} -\Delta u - \mu u^+ + \nu u^- = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has nontrivial solutions (even in the case of special domains Ω) .

(ii) If S is a linear completely continuous operator in the Hilbert space with a certain "good" cone (semiordering) , consider the mapping

$$T_{(\mu,\nu)} : u \longmapsto u - \mu S u^+ + \nu S u^- , \quad u \in H.$$

Is it true that if $T_{(\mu,\nu)} u = 0$ has only trivial solution then the Leray-Schauder degree of the mapping $T_{(\mu,\nu)}$ is nonzero if and only if $T_{(\mu,\nu)}$ is onto H ?

The answer is affirmative if (1) with $N = 1$ is considered, unknown in the case $N > 1$.

<u>3.2.</u> $\nu = \lambda_1$, $\mu = -\infty$

<u>3.2.1.</u> A typical example of this case is the Dirichlet problem

$$(17) \qquad \begin{cases} -u'' - \lambda u + e^u = f & \text{in } (0, \pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

with $\lambda = 1$. It is possible to prove that (17) has a weak solution for $f \in L_1(0,\pi)$ provided

$$\int_{\Omega} f(x) \sin x \, dx > 0 .$$

3.2.2. OPEN PROBLEM. The solvability of (17) with $\lambda > 1$ is an open problem. It is easy to see that there exists a right hand side f for which (17) is not solvable.

4. Rapid nonlinearities

4.1. Consider (1) with $N = 1$. If

$$\lim_{|\xi|\to\infty} \frac{g(\xi)}{\xi} = \infty \tag{18}$$

then (1) has infinitely many solutions (see [7]).

4.2. OPEN PROBLEM. The solvability of (1) with $N > 1$ under the assumption (18) , e.g. the weak solvability of

$$\begin{cases} -\Delta u - |u|^{\varepsilon} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if $\varepsilon > 0$ is sufficiently small, seems to be terra incognita.

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