BASIC REGULARITY OF THE MINIMA OF VARIATIONAL INTEGRALS

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The aim of this talk is to discuss the regularity of the minima of variational integrals:

$$(1) \qquad F(u;A) = \int_A f(x,u,Du)\,dx \quad .$$

For long time all the informations concerning this functional were obtained by means of its Euler equation. It was only recently that the so-called direct methods have established themselves as the via regia to deal with existence problems. A well-known result is the following: Let A be a bounded domain in $\mathbb{R}^n$ and let $f(x,u,p)$ be measurable in $x\in A$, continuous in $u\in\mathbb{R}^N$, convex in $p\in\mathbb{R}^{nN}$ and satisfy

$$(2) \qquad c|p|^m \leq f(x,u,p) \leq C(|p|^m + 1) \quad ; \; c>0 \;,\; m>1 \;.$$

Then the functional (1) has a minimum in the class of all functions in $W^{1,m}(A)$, taking prescribed values on ∂A.

The situation is different when we deal with the regularity of the minima. Here all the relevant results have as a common starting point the Euler equation of F in its weak form:

$$(3) \qquad \int_A \{f_p(x,u,Du)D\varphi + f_u(x,u,Du)\varphi\}\,dx = 0 \qquad \forall\varphi\in C_0^\infty(A).$$

Equation (3) is clearly essential when treating higher regularity; however its introduction at earlier stages might have some disadvantages, since it cannot be derived without suitable assumptions on the function f and on u.

In the first place, it is obviously necessary to suppose that f has partial derivatives with respect to u and p. Secondly, and most important, equation (3) is practically useless without additional hypotheses concerning the growth of f and of its derivatives. A "natural" behaviour is given by inequality (2) together with:

$$(4) \qquad |f_p(x,u,p)| \leq C(|p|^{m-1} + 1)$$

$$(5) \qquad |f_u(x,u,p)| \leq C(|p|^m + 1) \qquad ;$$

but these natural conditions do not imply regularity unless we suppose that u is bounded, and even small when $N > 1$.

If we want to avoid additional assumptions on u we have to replace (5) with some stronger condition. For instance, the inequality

$$(5') \qquad |f_u(x,u,p)| \leq C(|p|^{m-1} + 1)$$

will work, but it is quite innatural, unless f is of very special type.

Finally, the method based on the Euler equation does not distinguish between true minima and stationary points, or even solutions of elliptic equations that are not the Euler equation of a functional.

In conclusion, it seems preferable to prove the basic regularity results working directly with the functional F. Of course, regularity has different meaning in the scalar ($N=1$) and in the vector case ($N>1$). When $N=1$ we aim at the Hölder-continuity of the minima, following the way opened by the celebrated work by De Giorgi and its generalizations by Ladyzenskaya and Ural'ceva. In the general case, we cannot expect regularity everywhere, and our goal will be to show that the minima of F are in some Sobolev space $W^{1,q}$, $q > m$; a result originally proved by Boyarskii and Meyers. Moreover, we will look for the partial regularity of the minima.

Some attempts in this direction have been made by Ladyzenskaya and Ural'ceva who proved that any function minimizing F and taking bounded boundary values is bounded. For what concerns local results, i.e. independent of boundary values, we have a theorem by Morrey [5, Th. 4.3.1] valid when $m = n$, and some partial results obtained by Frehse [2] in the case $N=1$ and by Attouch and Sbordone [1] for $N>1$. The results that follow have been proved recently by Giaquinta and Giusti [3,4].

Theorem 1 Let f satisfy inequalities (2). Then:

(i) If $N=1$ every local minimum of the functional F is Hölder-continuous in A.

(ii) If $N>1$ every local minimum belongs to the Sobolev space $W^{1,q}_{loc}(A)$, for some $q>m$.

We recall that a local minimum of F is a function $u \in W^{1,m}_{loc}(A)$ such that for every $\varphi \in W^{1,m}(A)$ with $K = \mathrm{spt}\,\varphi \subset\subset A$ we have

$$F(u;K) \leq F(u+\varphi;K).$$

It is worth remarking that the conclusion of the theorem does not hold in general for extremals of F, even if we assume that f is convex in p (see [2]).

Further regularity results in the scalar case, as for instance the differentiability of the minima, require the use of the Euler equation and are quite standard. In the vector case, one may look for partial regularity, i.e. Hölder-continuity in some subset of A.

Some results in this direction can be proved for quadratic functionals:

$$(6) \qquad F(u;A) = \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_A A_{ij}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^j \, dx$$

when the coefficients $A_{ij}^{\alpha\beta}$ are bounded and uniformly continuous in $A \times \mathbb{R}^N$ and satisfy

$$(7) \qquad \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} A_{ij}^{\alpha\beta}(x,u)\, \xi_\alpha^i \xi_\beta^j \geq \lambda |\xi|^2 \quad ; \quad \lambda > 0.$$

We have:

Theorem 2 Let the hypotheses above be satisfied and let u be a local minimum for the functional F. Then there exists an open set $A_0 \subset A$ such that u is Hölder-continuous in A_0 and

$$H^{n-q}(A - A_0) = 0$$

for some $q > 2$, H^{n-q} denoting the $(n-q)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

In particular, the dimension of the singular set $A-A_o$ is strictly less than $n-2$.

The question may be raised whether the dimension of $A-A_o$ is at most $n-3$ (remember that the Hausdorff dimension is not necessarily an integer). The answer is affirmative in a special case at least.

Theorem 3 ([4]) *Let the coefficients* $A^{\alpha\beta}_{ij}$ *in (6) take the special form*

$$A^{\alpha\beta}_{ij}(x,u) = g_{ij}(x)\; G^{\alpha\beta}(x,u)$$

and let u *be a local minimum of* F. *Then* :

(i) *If* $n=3$, u *may have at most isolated singularities*.

(ii) *If* $n>3$, *the dimension of the singular set cannot exceed* $n-3$.

References.

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