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In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 264--267.

Persistent URL: http://dml.cz/dmlcz/702302

GENERALIZED SOMMERFELD HALF-PLANE PROBLEMS

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In NOBLE's book (1958) [7] the classical Sommerfeld half-plane diffraction problems are discussed. A one-dimensional Fouriertransformation with respect to the variable x parallel to the screen $\mathcal{S} := \{(x,y)\in\mathbb{R}^2 : x\geq 0, y=0\}$ is applied upon the scattered wave-function $\phi_{sc}(x,y)$ being a solution to the Helmholtz equation. In the cases of pure Dirichlet and Neumann conditions, respectively, the corresponding Fouriertransforms may be written for $y \gtrless 0$ as

$$\hat{\Phi}_{sc,D}(\lambda,y) = -\frac{1}{2}\gamma^{-1}\cdot\hat{J}_+(\lambda)\, e^{-|y|\gamma} \quad \text{and} \tag{1}$$

$$\hat{\Phi}_{sc,N}(\lambda,y) = \frac{1}{2}\hat{Q}_+(\lambda)\, e^{-|y|\gamma} \tag{2}$$

with the unilateral Fouriertransforms of the jumps $[\partial\hat{\Phi}_{sc}/\partial y]_0$ and $[\hat{\Phi}_{sc}]_0$ across the screen $\mathcal{S}$, respectively. Introducing additionally the unknown left unilateral Fouriertransforms $\hat{E}_-(\lambda) = \hat{\Phi}_-(\lambda,0)$ and $\hat{V}_-(\lambda) = \partial/\partial y\,\hat{\Phi}(\lambda,0)$ the following two scalar Wiener-Hopf functional equations hold for $-k_2\cos\Theta < \operatorname{Im}\lambda < k_2$ in case of an incident plane wave $\exp[ik(x\cos\Theta + y\sin\Theta)]$:

$$\hat{E}_-(\lambda) + \frac{1}{2}\gamma^{-1}\cdot\hat{J}_+(\lambda) = -[i\sqrt{2\pi}(\lambda+k\cos\Theta)]^{-1} \tag{3}$$

$$\hat{V}_-(\lambda) + \frac{1}{2}\gamma\cdot\hat{Q}_+(\lambda) = k\sin\Theta\cdot[\sqrt{2\pi}(\lambda+k\cos\Theta)]^{-1} \tag{4}$$

$\gamma := \sqrt{\lambda^2-k^2}$ with branch cuts extending from $k = k_1 + ik_2$ to $i\infty$ and $-k$ to $-i\infty$ and $\operatorname{Re}\gamma \geq 0$ in the strip $|\operatorname{Im}\lambda| < k_2$. By means of the classical scalar Wiener-Hopf technique the eqs. (3) and (4) are solved for $\hat{E}_-(\lambda)$, $\hat{J}_+(\lambda)$ and $\hat{V}_-(\lambda)$, $\hat{Q}_+(\lambda)$, respectively.

We generalize now to the *mixed Sommerfeld half-plane problem* where Dirichlet data are prescribed on the upper face $y = +0$ and Neumann data on the lower face $y = -0$ of the screen $\mathcal{S}$. Now the Fouriertransformation leads to the 2x2-Wiener-Hopf functional system

$$\gamma\cdot\hat{E}_-(\lambda) + \hat{V}_-(\lambda) + \hat{\Phi}'_+(\lambda,+0) = i\gamma\cdot[\sqrt{2\pi}(\lambda+k\cos\Theta)]^{-1} \tag{5a}$$

$$\gamma\cdot\hat{E}_-(\lambda) - \hat{V}_-(\lambda) + \gamma\cdot\hat{\Phi}_+(\lambda,-0) = k\sin\Theta[\sqrt{2\pi}(\lambda+k\cos\Theta)]^{-1} \tag{5b}$$

with the right unilateral Fouriertransforms of the unknown values $\partial/\partial y\,\phi(x,y)|_{y=+0}$ and $\phi(x,-0)$ for $x > 0$. After introducing the lower and upper complex half-plane holomorphic function-vectors

$$\hat{\Phi}_-(\lambda) := \begin{pmatrix} \sqrt{\lambda-k}\,\hat{E}_-(\lambda) \\ \hat{V}_-(\lambda)/\sqrt{\lambda-k} \end{pmatrix}, \quad \hat{\Phi}_+(\lambda) := -\frac{1}{2}\begin{pmatrix} \sqrt{\lambda+k}\cdot\hat{\Phi}(\lambda,-0) \\ \hat{\Phi}_+'(\lambda,+0)/\sqrt{\lambda+k} \end{pmatrix} \tag{6}$$

we arrive at the vectorial Wiener-Hopf equation

$$\hat{\Phi}_-(\lambda) := \begin{pmatrix} \sqrt{\frac{\lambda-k}{\lambda+k}} & , & 1 \\ -1 & , & \sqrt{\frac{\lambda+k}{\lambda-k}} \end{pmatrix} \cdot \hat{\Phi}_+(\lambda) + \hat{\vec{r}}(\lambda) \tag{7}$$

with the $\vec{r}(\lambda)$ containing the transformed boundary data. The 2x2-matrix has been factorized into $K_-(\lambda)\cdot[K_+(\lambda)]^{-1}$ explicitly by A. D. RAWLINS (1980) [10] and the author ([1981]) [6]. The matrix elements are given by

$$K_{11}^-(\lambda) = i\sqrt{\lambda-k}\cdot[\sqrt{2k} + i\sqrt{\lambda-k}]^{-1/2} \tag{8a}$$

$$K_{12}^-(\lambda) = i\cdot[\sqrt{2k} + i\sqrt{\lambda-k}]^{-1/2} \tag{8b}$$

$$K_{21}^-(\lambda) = -i\cdot[\sqrt{2k} + i\sqrt{\lambda-k}\,]^{1/2} \tag{8c}$$

$$K_{22}^-(\lambda) = i\cdot[\sqrt{2k} + i\sqrt{\lambda-k}\,]^{1/2}/\sqrt{\lambda-k} \tag{8d}$$

being holomorphic for $\text{Im}\,\lambda < k_2$ and

$$K_{11}^+(\lambda) = \frac{i}{2}\{[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2} + [\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2}\} \tag{9a}$$

$$K_{12}^+(\lambda) = \frac{i}{2}\{[\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2} - [\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}\}/\sqrt{\lambda-k} \tag{9b}$$

$$K_{21}^+(\lambda) = \frac{i}{2}\{[\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2} - [\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2}\}\sqrt{\frac{\lambda-k}{\lambda+k}} \tag{9c}$$

$$K_{22}^+(\lambda) = \frac{i}{2}\{[\sqrt{2k}+i\sqrt{\lambda-k}]^{1/2} + [\sqrt{2k}-i\sqrt{\lambda-k}]^{1/2}\}/\sqrt{\lambda+k} \tag{9d}$$

being holomorphic for $\text{Im}\,\lambda > -k_2$.

After multiplication of eq. (7) by $[K_-(\lambda)]^{-1}$ and splitting $\hat{\vec{s}}(\lambda) = [K_-(\lambda)]^{-1}\hat{\vec{r}}(\lambda)$ additively into $\hat{\vec{s}}_+(\lambda) + \hat{\vec{s}}_-(\lambda)$ the unknown vectors may be represented as

$$\hat{\Phi}_\pm(\lambda) = K_\pm(\lambda)\hat{\vec{s}}_\pm(\lambda) \tag{10}$$

from which an explicit formula for $\hat{\Phi}(\lambda,y)$ may be derived. Generalizations to different impedance boundary conditions on the two faces where treated e. g. by A. HURD (1976) [4], A. D. RAWLINS (1975) [8].

Here we generalize the Sommerfeld problems to the cases of two parallel semi-infinite screens at a distance 2a and to a periodic system with gap-width a. The classical cases with pure boundary conditions have been treated before, e. g. by A. E. HEINS in (1948) [2] and (1946/50) [1] and the author (1970) [5] but those with mixed Dirichlet-Neumann conditions seem to be new. Applying the Fouriertransformation leads to three regions, viz. $y>a$, $|y|<a$, $y<-a$ for $\hat{\Phi}(\lambda,y)$. Now four unknown unilateral right and left Fouriertransforms arise corresponding to the four half-lines $y = \pm a$, $x \gtrless 0$: $\hat{E}_{\pm a}(\lambda)$, $\hat{V}_{\pm}a(\lambda)$, being holomorphic in lower λ-half-planes, and $\hat{\Phi}_+(\lambda,\pm(a-o))$, $\partial/\partial y\hat{\Phi}_+(\lambda,\pm(a+o))$, being holomorphic in upper λ-half-planes, respectively. After suitable combinations of the resulting equations in the λ-domain one arrives at two systems of 2x2-Wiener-Hopf functional systems

$$\begin{matrix}\hat{\Phi}_-(\lambda)\\ \text{or}\\ \hat{\Psi}_-(\lambda)\end{matrix} + \frac{1}{2}\begin{pmatrix}(1\mp e^{-2a\gamma})\sqrt{\frac{\lambda-k}{\lambda+k}} , & 1 \pm e^{-2a\gamma}\\ -(1\pm e^{-2a\gamma}) , & (1\mp e^{-2a\gamma})\sqrt{\frac{\lambda+k}{\lambda-k}}\end{pmatrix}\begin{matrix}\hat{\Phi}_+(\lambda)\\ \text{or}\\ \hat{\Psi}_+(\lambda)\end{matrix} = \begin{matrix}\hat{s}^{(+)}(\lambda)\\ \text{or}\\ \hat{s}^{(-)}(\lambda)\end{matrix} \tag{11}$$

with known transforms $\hat{s}^{(+)}(\lambda)$ and $\hat{s}^{(-)}(\lambda)$. An explicit factorization of these 2x2-matrices $K^{(\pm)}(\lambda;a)$ is not yet known!

By a similar procedure one can reduce the mixed boundary value problem for $\Phi_{sc}(x,y)$ solving $(\Delta+k^2)\Phi_{sc} = 0$ outside the stack of plates $\bigcup_{n=-\infty}^{\infty} \mathcal{A}_n$ where $\mathcal{A}_n := \{(x,y)\in\mathbb{R}^2 : x \geq 0,\ y=n\cdot a\}$ with the underlying boundary conditions $\Phi_{sc}(x,na+o) = -\exp[ik(x\cos\theta + na\sin\theta)]$ and $\partial/\partial y\ \Phi_{sc}(x,na-o) = -ik\sin\theta\cdot$ $\cdot\exp[ik(x\cos\theta + na\sin\theta)]$ for $x > 0$, $n\in\mathbb{Z}$ to a Wiener-Hopf-(2x2)-functional system making use of the quasi-periodic boundary data with respect to y. Introducing $\hat{\vec{\Phi}}_-(\lambda)$ and $\hat{\vec{\Phi}}_+(\lambda)$ similar to eq. (6) we obtain

$$\hat{\vec{\Phi}}_-(\lambda) + K(\lambda;a,\theta)\hat{\vec{\Phi}}_+(\lambda) = [M(\lambda;a,\theta)]^{-1}\cdot\vec{r}(\lambda) \tag{12}$$

with the following two function matrices having elements

$$K_{11}(\lambda;a,\theta) := \frac{1}{2}\sqrt{\frac{\lambda-k}{\lambda+k}}\, e^{-iak\sin\theta}[1-\frac{i\cdot\sin(ak\sin\theta)}{\cosh(a\gamma)-\cos(ak\sin\theta)}] \tag{13a}$$

$$K_{12}(\lambda;a,\theta) := \frac{1}{4}[\coth\frac{a}{2}(\gamma-ik\sin\theta) + \coth\frac{a}{2}(\gamma+ik\sin\theta)] \tag{13b}$$

$$K_{21}(\lambda;a,\theta) := \frac{1}{4}[\coth\frac{a}{2}(\gamma-ik\sin\theta) + \coth\frac{a}{2}(\gamma+ik\sin\theta)] \tag{13c}$$

$$K_{22}(\lambda;a,\theta) := \frac{1}{2}\sqrt{\frac{\lambda+k}{\lambda-k}}[1+\frac{i\cdot\sin(ak\sin\theta)}{\cosh(a\gamma)-\cos(ak\sin\theta)}] \tag{13d}$$

and

$$M_{11}(\lambda;a,\theta) := 1-e^{iak\sin\theta}\cdot\cosh(a\gamma) \tag{14a}$$

$$M_{12}(\lambda;a,\theta) := -\sqrt{\frac{\lambda-k}{\lambda+k}}\, e^{iak\sin\theta}\cdot\sinh(a\gamma) \tag{14b}$$

$$M_{21}(\lambda;a,\theta) := e^{iak\sin\theta}\cdot\sinh(a\gamma) \tag{14c}$$

$$M_{22}(\lambda;a,\theta) := -\sqrt{\frac{\lambda-k}{\lambda+k}}\,[1-e^{iak\sin\theta}\cdot\cosh(a\gamma)]. \tag{14d}$$

An explicit factorization of $K(\lambda;a,\theta)$ is not known up to now.

Similar systems of Wiener-Hopf functional equations in the Fourier transform plane may be derived for mixed impedance boundary conditions on the different faces of the plates of the infinite periodic system or for semi-infinite circular tubes (cf. e. g. A. D. RAWLINS (1978)[9]!). A detailed version of the material presented here may be found in the author's lectures held at the Stefan Banach International Mathematical Center, Warsaw, Spring term 1981 [6].

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