EQUADIFF 5

Nguyen Thanh Bang

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In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 280--283.

Persistent URL: http://dml.cz/dmlcz/702306

MULTIPLE PICARD'S METHOD FOR THE STIFF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

Nguyen Thanh Bang
Hanoi, Warsaw/ SR Vietnam, Poland

Abstract. A combination of Picard's method developed by the author in Refs. [1,2] and a time-decomposition technique is proposed to solve the stiff nonlinear two-point boundary value problems in case where the integration interval is large.

Some constructive sufficient conditions for convergence of the combination method are presented.

1. Introduction. It is well known that a broad class of optimal control problems, the investigation of which, due to Pontryagin's maximum principle, reduces to a nonlinear two-point boundary value problem of the form

$$\frac{dx}{dt}=A(t)x+B(t)\psi+f(x,\psi,t),\ \frac{d\psi}{dt}=Q(t)x-A'(t)\psi+g(x,\psi,t) \qquad (1.1)$$

for $t\in[t_0,t_f]$, subject to the boundary conditions

$$x(t_0)=a_0\ ,\ Mx(t_f)+N\psi(t_f)=a_f \qquad (1.2)$$

where x and ψ are n-dimensional functions of time t, $A(t)$, $B(t)$ and $Q(t)$ are known $(n\times n)$-dimensional matrices, all elements of which are assumed to be continuous on the integration interval $t_0\le t\le t_f$, $f(x,\psi,t)$ and $g(x,\psi,t)$ are assumed to be continuous in all arguments in some closed domain of the (x,ψ,t)-space, M and N are known $(n\times n)$-dimensional constant matrices, t_0 is the initial time, t_f is the fixed terminal time, a_0 and a_f are given n-dimensional vectors.

Here, as elsewhere, the prime denotes the matrix transposition.

The problem is to find the functions

$$x=x(t)\ ,\ \psi=\psi(t)\ ,\ t_0\le t\le t_f \qquad (1.3)$$

which solve Eqs. (1.1) subject to the boundary conditions (1.2).

Eqs. (1.1) are known often to have a stiff structure, i.e. some of the particular solutions increase and others descrease rapidly as the independent variable changes. The exponential growth of some components of a solution might lead to numerical difficulties, especially when the integration interval is large. Because of this exponential growth, overflow can occur in computer. Even when overflow does not occur, in the last case a lot of known approximation methods [1-6] often fail to offer a satisfactory solution because of numerical errors.

To overcome these difficulties and to provide the convergence of the iterative process, a multipoint approach to the two-point boundary value problems was proposed in Ref.[7] and then developed by many authors [5,8-11].

Multiple methods proposed in Refs.[5,8-10] require to determine all the boundary values at once, so that they must take the inverse of $n(2m-1)\times n(2m-1)$-dimensional matrix, where m is a number of subintervals. The technique proposed in Ref.[11] needs only to take the inverse of $n(m-1)\times n(m-1)$-dimensional matrix.

In this paper, the multiple Picard's approach to the two-point boundary value problem is developed. The present technique requires to take only the inverse of $(n\times n)$-dimensional matrix, so that one is preferable, from computational point view, to the others known from literature [5,8-11].

2. <u>Multiple Picard's Method</u>. Let ${}^{j}x(t)$ and ${}^{j}\psi(t)$ denote j-th Picard's iterate for the function $x(t)$ and $\psi(t)$ which solve Eqs.(1.1) subject to (1.2). If a number $T=t_f-t_0$ is large, then, following Refs.[7-11], we divide the overall integration interval $t_0\leq t\leq t_f$ into m subintervals by m-1 time points $t_1, t_2, \ldots, t_{m-1}$ which are intermediate between the initial time t_0 and the final time t_f and such that the numbers $\tau_i=t_i-t_{i-1}$, i=1,2,...,m $(t_m=t_f)$ are sufficiently small. These subintervals are numbered as follows: subinterval i, $t_{i-1}\leq t\leq t_i$, i=1,2,...,m.

With the above conventions, let ${}^{j}x_i(t)$ and ${}^{j}\psi_i(t)$ denote the portions of the functions ${}^{j}x(t)$ and ${}^{j}\psi(t)$ pertaining to the i-th subinterval.

Then, it is clear from the results obtained in Ref.[2] that the functions ${}^{j}x_i(t)$ and ${}^{j}\psi_i(t)$ must satisfy the following equations

$$\frac{d[{}^{j}x_i]}{dt}=A(t)[{}^{j}x_i]+B(t)[{}^{j}\psi_i]+\delta_j f({}^{j-1}x(t),{}^{j-1}\psi(t),t),\quad \frac{d[{}^{j}\psi_i]}{dt}=Q(t)[{}^{j}x_i]-A'(t)[{}^{j}\psi_i]+\delta_j g({}^{j-1}x(t),{}^{j-1}\psi(t),t) \tag{2.1}$$

for $t\in[t_{i-1},t_i]$, subject to the boundary conditions

$${}^{j}x_1(t_0)=a_0\,,\quad M[{}^{j}x_m(t_m)]+N[{}^{j}\psi_m(t_m)]=a_f\,. \tag{2.2}$$

In addition, at the interface between a subinterval and the next, the following continuity conditions must be satisfied:

$${}^{j}x_i(t_i)={}^{j}a_i:={}^{j}x_{i+1}(t_i)\,,\ {}^{j}\psi_i(t_i)={}^{j}\lambda_i:={}^{j}\psi_{i+1}(t_i),\ i=1,2,\ldots,m-1. \tag{2.3}$$

The functions ${}^{j-1}x(\cdot)$ and ${}^{j-1}\psi(\cdot)$ on the right-hand side of Eqs.(2.1) are assumed, as a rule, to be already known and the numbers δ_j, j=0,1,2,...are defined as follows

$$\delta_0=0\,,\ \delta_j=1\,,\ \forall j\geq 1. \tag{2.4}$$

Let $H(t,\xi)$ be the $(2n\times 2n)$-dimensional transition matrix for the Eqs.(2.1) subject to $\delta_j=0$ and we partition this matrix into four $(n\times n)$-dimensional matrices as follows

$$H(t,\xi)=\begin{bmatrix} H_{11}(t,\xi) & H_{12}(t,\xi) \\ H_{21}(t,\xi) & H_{22}(t,\xi) \end{bmatrix}$$

Now, let ${}^j\lambda_0$ denotes the missing initial condition for ${}^j\psi_1(t)$ at $t=t_0$, we have

<u>Subinterval 1, $t_0 \le t \le t_1$:</u>

$$\begin{bmatrix} {}^jx_1(t) \\ {}^j\psi_1(t) \end{bmatrix}=\begin{bmatrix} H_{11}(t,t_0) & H_{12}(t,t_0) \\ H_{21}(t,t_0) & H_{22}(t,t_0) \end{bmatrix}\begin{bmatrix} 0 \\ {}^j\lambda_0 \end{bmatrix}+\begin{bmatrix} {}^j\tilde{x}_1(t) \\ {}^j\tilde{\psi}_1(t) \end{bmatrix} \qquad (2.5)$$

<u>Subinterval i, $t_{i-1} \le t \le t_i$, i=2,3,...,m:</u>

$$\begin{bmatrix} {}^jx_i(t) \\ {}^j\psi_i(t) \end{bmatrix}=\begin{bmatrix} H_{11}(t,t_{i-1}) & H_{12}(t,t_{i-1}) \\ H_{21}(t,t_{i-1}) & H_{22}(t,t_{i-1}) \end{bmatrix}\begin{bmatrix} {}^ja_{i-1} \\ {}^j\lambda_{i-1} \end{bmatrix}+\begin{bmatrix} {}^j\tilde{x}_i(t) \\ {}^j\tilde{\psi}_i(t) \end{bmatrix} \qquad (2.6)$$

where the functions ${}^j\tilde{x}_1(t)$ and ${}^j\tilde{\psi}_1(t)$ are the solution to Eqs.(2.1) for $t\in[t_0,t_1]$, subject to the initial conditions

$${}^j\tilde{x}_1(t_0)=a_0\ ,\ {}^j\tilde{\psi}_1(t_0)=0 \qquad (2.7)$$

and the functions ${}^j\tilde{x}_i(t)$ and ${}^j\tilde{\psi}_i(t)$, i=2,3,...,m, are the solution to Eqs. (2.1) for $t\in[t_{i-1},t_i]$, subject to the initial conditions

$${}^j\tilde{x}_i(t_{i-1})={}^j\tilde{\psi}_i(t_{i-1})=0. \qquad (2.8)$$

The problem now consists in determining the vectors ${}^j\lambda_0$, ${}^ja_1$, ${}^j\lambda_1$ ..., ${}^ja_{m-1}$ and ${}^j\lambda_{m-1}$ under which the terminal condition (2.2) and the continuity conditions (2.3) are satisfied.

It turns out that the following result holds.

<u>THEOREM 1</u>. If the matrices $H_{12}(t_1,t_0)$ and $\Gamma(t_m,t_1,t_0)$ are nonsingular, then

1/ n-dimensional vector ${}^ja_1$ representing the value of the function ${}^jx(t)$ of j-th Picard's iterate at the intermediate time $t=t_1$ is uniquely defined by algebraic equation

$$\Gamma(t_m,t_1,t_0)[{}^ja_1]={}^j\tilde{a}_f \qquad (2.9)$$

where

$$\Gamma(t_m,t_1,t_0)=F(t_m,t_1)+G(t_m,t_1)H_{22}(t_1,t_0)H_{12}^{-1}(t_1,t_0), \qquad (2.10)$$

$${}^j\tilde{a}_f=a_f-\sum_{i=2}^{m}\{F(t_m,t_i)[{}^j\tilde{x}_i(t_i)]+G(t_m,t_i)[{}^j\tilde{\psi}_i(t_i)]\}+G(t_m,t_1)\{H_{22}(t_1,t_0)H_{12}^{-1}(t_1,t_0)[{}^j\tilde{x}_1(t_1)]-[{}^j\tilde{\psi}_1(t_1)]\}$$

$$F(t_m,t_i)=MH_{11}(t_m,t_i)+NH_{21}(t_m,t_i),\ G(t_m,t_i)=MH_{12}(t_m,t_i)+NH_{22}(t_m,t_i).$$

2/ The remainning $2n(m-1)$ boundary conditions ${}^j\lambda_0$, ${}^j\lambda_1$, ${}^ja_2$, ${}^j\lambda_2,\dots,{}^ja_{m-1}$ and ${}^j\lambda_{m-1}$ can be then evaluated by formulae

$${}^j\lambda_0=H_{12}^{-1}(t_1,t_0)[{}^ja_1-{}^j\tilde{x}_1(t_1)] \qquad (2.11)$$

$${}^j\lambda_1=H_{22}(t_1,t_0)[{}^j\lambda_0]+{}^j\tilde{\psi}_1(t_1) \qquad (2.12)$$

and by the following recurrent relations

$$\begin{bmatrix} {}^ja_i \\ {}^j\lambda_i \end{bmatrix}=H(t_i,t_{i-1})\begin{bmatrix} {}^ja_{i-1} \\ {}^j\lambda_{i-1} \end{bmatrix}+\begin{bmatrix} {}^j\tilde{x}_i(t_i) \\ {}^j\tilde{\psi}_i(t_i) \end{bmatrix}\quad ,\ i=2,3,\dots,m-1. \qquad (2.13)$$

3. <u>On the Convergence of the Multiple Picard's Method</u>.

<u>THEOREM 2</u>. Assume

1/ The matrices $H_{12}(t_1,t_0)$ and $\Gamma(t_m,t_1,t_0)$ defined by (2.10) are nonsingular.

2/The function $h(z,t)=[f'(z,t),g'(z,t)]'$ is continuous with respect to all arguments in certain closed domain D of the (z,t)-space determined by the expression

$$D=\{(z,t):\ |z|\leq r\ ,\ t_0\leq t\leq t_f\} \tag{3.1}$$

where $z=[x',\psi']'$ and the norm of a matrix z is denoted by $|z|$.

3/In the domain D function $h(z,t)$ is Lipschitzian with respect to z with the Lipschitz's constant L .

4/The numbers a_1 and a_2 connected with $|a_0|$ and $|a_f|$ by formulae

$$a_1=1+\rho h_{12}^{-}(1+\nu g\Gamma^{-})|a_0|+h_{12}^{-}\Gamma^{-}|a_f|$$
$$a_2=\rho[1+(1+\nu g\Gamma^{-})h_{22}h_{12}^{-}+\nu g\Gamma^{-}]|a_0|+\nu\Gamma^{-}|a_f|$$

satisfy the condition

$$\max(a_1,a_2)<\frac{1}{\rho}r \tag{3.2}$$

where $\rho=\max\limits_{t_0\leq t\leq \xi\leq t_f}|H(t,\xi)|$, $h_{12}^{-}=|H_{12}^{-1}(t_1,t_0)|$, $g=|G(t_m,t_1)|$,
$h_{22}=|H_{22}(t_1,t_0)|$, $\Gamma^{-}=|\Gamma^{-1}(t_m,t_1,t_0)|$, $\nu=1+h_{22}h_{12}^{-}$

5/ $0<\tau<\tau^{*}=(1/\rho)\min\{r-\rho a_1/\alpha_m H,\ r-\rho a_2/\beta_m H\ ,\ 1/\alpha_m L,\ 1/\beta_m L\}$ (3.3)

where $\tau=\max\limits_{1\leq i\leq m}\tau_i$, $\tau_i=t_i-t_{i-1}$, $\vartheta_m=1+\nu g\Gamma^{-}+(m-1)\rho\sigma\Gamma^{-}$, $\sigma=\max(|M|,|N|)$,
$\alpha_m=1+\vartheta_m\rho h_{12}^{-}$, $\beta_m=1+[(m-2)+\nu\vartheta_m]\rho$, $H=\max|h(z,t)|,\forall(z,t)\in D$.

Then,the multiple Picard's method presented above for solving the problem (1.1) and (1.2) is convergent.

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