NONOSCILLATION THEOREMS FOR A CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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We consider the neutral functional differential equation

$$(A_\sigma) \qquad \frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] + \sigma f(t, x(g(t))) = 0,$$

where $n \geq 2$, $\sigma = +1$ or -1, $\lambda(\neq 1)$ and τ are positive constants, and $g : [t_0,\infty) \to \mathbb{R}$ and $f : [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $\lim_{t\to\infty} g(t) = \infty$, $uf(t, u) \geq 0$ for $(t,u) \in [t_0,\infty) \times \mathbb{R}$ and $f(t, u)$ is nondecreasing in u for each fixed $t \geq t_0$.

It is easy to see that the following four types of asymptotic behavior at infinity are possible for nonoscillatory solutions $x(t)$ of (A_σ):

(I) $\quad \lim_{t\to\infty} x(t) = 0$;

$(\mathrm{II})_k$ $\quad \lim_{t\to\infty} x(t)/t^k = \text{const} \neq 0$ for some $k \in \{0, 1, \ldots, n-1\}$;

$(\mathrm{III})_\ell$ $\quad \lim_{t\to\infty} x(t)/t^\ell = 0$ and $\lim_{t\to\infty} x(t)/t^{\ell-1} = \infty$ or $-\infty$ for some $\ell \in \{1, 2, \ldots, n-1\}$;

(IV) $\quad \lim_{t\to\infty} x(t)/t^{n-1} = \infty$ or $-\infty$.

A natural question then arises: Is it possible to characterize the classes of nonoscillatory solutions of (A_σ) having the asymptotic behavior (I), $(\mathrm{II})_k$, $(\mathrm{III})_\ell$ and (IV), respectively?

Our objective here is give a partial answer to the above question. Our main results are as follows:

THEOREM 1. Let $0 < \lambda < 1$. If there exist constants $\mu \in (0, \lambda)$ and $a \neq 0$ such that

$$(1) \qquad \int^\infty t^{n-1}\mu^{-t/\tau}|f(t, a\lambda^{g(t)/\tau})|dt < \infty,$$

then equation (A_σ) has a decaying nonoscillatory solution $x(t)$ with the property

$$(2) \qquad x(t) = \text{const}\cdot\lambda^{t/\tau} + o(\lambda^{t/\tau}) \text{ as } t \to \infty.$$

THEOREM 2. Equation (A_σ) has a nonoscillatory solution $x(t)$ satisfying

$$(3) \qquad \lim_{t\to\infty} x(t)/t^k = \text{const} \neq 0 \text{ for some } k \in \{0, 1, \ldots, n-1\}$$

if $0 < \lambda < 1$ and

$$(4) \qquad \int^\infty t^{n-k-1}|f(t, a[g(t)]^k)|dt < \infty \text{ for some } a \neq 0$$

or if $\lambda > 1$ and

(5) $$\int^{\infty} t^{n-k-1}|f(t, a[g^*(t)]^k)|dt < \infty \text{ for some } a \neq 0,$$

where $g^*(t) = \max\{g(t), t\}$.

THEOREM 3. (i) Let $0 < \lambda < 1$ and let $\ell \in \{1, 2, \ldots, n-1\}$ be such that $(-1)^{n-\ell-1}\sigma = 1$. Equation (A_σ) has a nonoscillatory solution $x(t)$ satisfying

(6) $$\lim_{t\to\infty} x(t)/t^{\ell} = 0 \text{ and } \lim_{t\to\infty} x(t)/t^{\ell-1} = \infty \text{ or } -\infty.$$

if

(7) $$\int^{\infty} t^{n-\ell-1}|f(t, a[g(t)]^{\ell})|dt < \infty \text{ for some } a \neq 0$$

and

(8) $$\int^{\infty} t^{n-\ell}|f(t, b[g(t)]^{\ell-1})|dt = \infty \text{ for all } b \neq 0.$$

(ii) Let $\lambda > 1$ and let $\ell \in \{1, 2, \ldots, n-1\}$ be such that $(-1)^{n-\ell-1}\sigma = -1$. Equation (A_σ) has a nonoscillatory solution $x(t)$ satisfying (6) if

(9) $$\int^{\infty} t^{n-\ell-1}|f(t, a[g^*(t)]^{\ell})|dt < \infty \text{ for some } a \neq 0$$

and

(10) $$\int^{\infty} t^{n-\ell}|f(t, b[g(t)]^{\ell-1})|dt = \infty \text{ for all } b \neq 0.$$

THEOREM 4. Let $\lambda > 1$. Equation (A_σ) has a growing nonoscillatory solution $x(t)$ such that

(11) $$x(t) = \text{const}\cdot\lambda^{t/\tau} + o(\lambda^{t/\tau}) \text{ as } t \to \infty$$

if either

(12) $$\int^{\infty} t^{n-1}|f(t, a\lambda^{g^*(t)/\tau})|dt < \infty \text{ for some } a \neq 0$$

or

(13) $$\int^{\infty} \mu^{-t/\tau}|f(t, a\lambda^{g^*(t)/\tau})|dt < \infty \text{ for some } \mu \in (1, \lambda) \text{ and } a \neq 0.$$

These theorems are proved by solving, via the Schauder-Tychonoff fixed point theorem, "integral-difference" equations of the types

(14) $$x(t) - \lambda x(t-\tau) = c + (-1)^{n-1}\sigma\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}f(s, x(g(s)))ds,$$

(15) $$x(t) - \lambda x(t-\tau) = c(t-T)^k/k! \quad [\text{or } c(t-T)^{k-1}/(k-1)!]$$
$$+ (-1)^{n-k-1}\sigma\int_T^t\frac{(t-s)^{k-1}}{(k-1)!}\int_s^{\infty}\frac{(r-s)^{n-k-1}}{(n-k-1)!}f(r, x(g(r)))drds,$$

(16) $$x(t) - \lambda x(t-\tau) = -\sigma\int_T^t\frac{(t-s)^{n-1}}{(n-1)!}f(s, x(g(s)))ds,$$

c and T being suitably chosen constants. For example, the proof of Theorem 4 under the condition (13) proceeds as follows. Suppose that $a > 0$ in (13). Let $\nu \in (\mu, \lambda)$

be fixed and choose $c > 0$ and $T > t_0$ so that $2\lambda c/(\lambda - \nu) \leq a$, $T_0 = \min\{T - \tau, \inf_{t \geq T} g(t)\} \geq t_0$, $t^{n-1}\mu^{t/\tau} \leq \nu^{t/\tau}$ for $t \geq T$, and

$$\int_T^\infty \mu^{-t/\tau} f(t, a\lambda^{g^*(t)/\tau})dt \leq c.$$

Define the set $X \subset C[T_0,\infty)$ and the mapping $F : X \to C[T_0,\infty)$ by

(17) $$X = \{x \in C[T_0,\infty) : |x(t)| \leq c\nu^{t/\tau} \text{ for } t \geq T_0\}$$

and

(18) $$\begin{cases} Fx(t) = -\sigma\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, \hat{x}(g(s)))ds, & t \geq T, \\ Fx(t) = 0, & T_0 \leq t \leq T, \end{cases}$$

where $\hat{x} : [T_0,\infty) \to \mathbb{R}$ is given by

(19) $$\begin{cases} \hat{x}(t) = \frac{\lambda c}{\lambda - \nu}\lambda^{t/\tau} - \sum_{i=1}^\infty \lambda^{-i}x(t + i\tau), & t \geq T, \\ \hat{x}(t) = \hat{x}(T), & T_0 \leq t \leq T. \end{cases}$$

It can be shown that F maps X continuously into a compact subset of X, so that there exists a fixed element $\xi \in X$ of F by the Schauder-Tychonoff theorem. Since the function $\hat{\xi}(t)$ satisfies $\hat{\xi}(t) - \lambda\hat{\xi}(t-\tau) = \xi(t)$ for $t \geq T$, it turns out that $\hat{\xi}(t)$ is a solution of (16), and hence a solution of (A_σ) on $[T,\infty)$. That $\hat{\xi}(t)$ grows like a constant multiple of $\lambda^{t/\tau}$ as $t \to \infty$ is an immediate consequence of its definition (19). Note that $\hat{\xi}(t)$ satisfies $\hat{\xi}(t) - \lambda\hat{\xi}(t-\tau) \to -\infty$ as $t \to \infty$.

If, on the other hand, (12) is satisfied for some $a > 0$, then the desired solution of (A_σ) in Theorem 4 is obtained as a solution $\tilde{\eta}(t)$ of the integral-difference equation (14) with $c = 0$. To see this it suffices to choose $c > 0$ and $T \geq t_0$ so that $2\lambda c/(\lambda - 1) \leq a$, $T_0 = \min\{T - \tau, \inf_{t \geq T} g(t)\} \geq t_0$ and $\int_T^\infty t^{n-1} f(t, a\lambda^{g^*(t)/\tau})dt \leq c$, and then to apply the Schauder-Tychonoff theorem to the mapping

(20) $$\begin{cases} Gy(t) = (-1)^{n-1}\sigma\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \tilde{y}(g(s)))ds, & t \geq T, \\ Gy(t) = Gy(T), & T_0 \leq t \leq T \end{cases}$$

defined on the set $Y = \{y \in C[T_0,\infty) : |y(t)| \leq c \text{ for } t \geq T_0\}$, where

(21) $$\begin{cases} \tilde{y}(t) = \frac{\lambda c}{\lambda - 1}\lambda^{t/\tau} - \sum_{i=1}^\infty \lambda^{-i}y(t + i\tau), & t \geq T, \\ \tilde{y}(t) = \tilde{y}(T), & T_0 \leq t \leq T. \end{cases}$$

It is clear that $\tilde{\eta}(t) - \lambda\tilde{\eta}(t-\tau) \to 0$ as $t \to \infty$. Since (12) implies (13), the condition (12) guarantees the existence of two different types of exponentially growing solutions $\hat{\xi}(t)$ and $\tilde{\eta}(t)$ such that $\lim_{t\to\infty} |\hat{\xi}(t) - \lambda\hat{\xi}(t-\tau)| = \infty$ and $\lim_{t\to\infty} [\tilde{\eta}(t) - \lambda\tilde{\eta}(t-\tau)] = 0$, respectively.

It would be natural to expect that analogues of the above theorems hold for the "companion" equation

$$(B_\sigma) \qquad \frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] + \sigma f(t, x(g(t))) = 0.$$

However, proving this conjecture does not seem to be an easy task; so far we have been able to prove only the following result which is an analogue of Theorem 2.

THEOREM 5. Let $\lambda > 0$, $\lambda \neq 1$ and $k \in \{0, 1, \ldots, n-1\}$. If condition (4) holds for some $a \neq 0$, then equation (B_σ) has a nonoscillatory solution $x(t)$ such that

$$0 < \liminf_{t\to\infty} |x(t)|/t^k, \ \limsup_{t\to\infty} |x(t)|/t^k < \infty.$$

There is no essential difficulty in extending the above results to more general equations of the form

$$\frac{d^n}{dt^n}[x(t) \pm \lambda(t)x(\tau(t))] + \sigma f(t, x(g_1(t)), \ldots, x(g_N(t))) = 0,$$

where $\lambda(t)$, $\tau(t)$ and $g_i(t)$, $1 \le i \le N$, are continuous functions on $[t_0,\infty)$ such that $\lambda(t)$ is positive and bounded, $\tau(t)$ is strictly increasing, $\lim_{t\to\infty} \tau(t) = \infty$ and $\lim_{t\to\infty} g_i(t) = \infty$, $1 \le i \le N$.

A systematic study of the existence and asymptotic behavior of nonoscillatory solutions of neutral functional differential equations was initiated by Ruan [4] and followed by Jaroš and Kusano [2, 3]. For a result ensuring the existence of decaying nonoscillatory solutions we refer to Gopalsamy [1].

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