ON THE NODAL SET OF EIGENFUNCTIONS

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In this note we are interested in qualitative properties of the eigenfunctions of the equation

$$(1)\qquad \begin{aligned} -\Delta v + q(x)v &= \lambda v \quad , \quad \text{in} \quad \Omega \subset R^2 \\ v &= 0 \quad , \quad \text{on} \quad \partial\Omega \quad , \end{aligned}$$

where $\Omega \subset R^2$ is a bounded and smooth domain, and $q \in L^\infty(\Omega)$.

It is wll-known that in one dimension the n-th eigenfunction v_n of the Sturm-Liouville eigenvalue problem

$$(2)\qquad -v'' + q(x)v = v \;\text{ in } (o,1), \;\; v(0) = v(1) = 0$$

has exactly n-1 nodes (i.e. nondegenerate zeroes).

In two (and higher) dimensions the situation is more complicated and relatively little is known. Let $Z_n = \{x \in \Omega;\; v_n(x) = 0\}$ denote the nodal set of the n-th eigenfunction v_n of (1), and denote by k_n the number of connected components of $\Omega \setminus Z_n$. Note that the stated result for one dimension says that $k_n = n$, for all $n \in N$. In two (and higher) dimensions one has only upper estimates for k_n. The Courant nodal domain theorem [3,4] states that $k_n \leq n$, for all $n \in N$. Furthermore, by a result of Pleijel [8] one has the asymptotic estimate

$$(3)\qquad \limsup_{n \to \infty} \frac{k_n}{n} \leq \frac{4}{j^2} < 0.7 \;,$$

where j denotes the smallest zero of the 0-th Bessel function. This implies that $k_n = n$ can occur only finitely many times. Relation (3) would still leave room for a lower estimate of the form $k_n \geq \alpha n$, $\alpha \in (0,4/j^2)$, but Stern [9] has given examples where $k_n = 2$ occurs infinitely many times.

Since the one-dimensional result gives a count of the nodal points, another possible generalization to higher dimensions could be a measure of the "size" of the nodal set. In fact, for $q \equiv 0$ Brüning and Gromes [2] show that if Ω is simply connected with area F and circumference U, then the length l_n of the nodal set of the n-th eigenfunction satisfies

$$l_n \geq \frac{F\sqrt{\lambda_n}}{2j} - \frac{U}{2} \;,$$

where j denotes again the smallest zero of the 0-th Besselfunction. Complementary, Donelly and Fefferman [5] have shown an upper bound $l_n \leq c\sqrt{\lambda_n}$, for all $n \in N$, (both results are valid for compact Riemannian manifolds of dimension two, [1]).

However, the following result shows that such estimates do not hold independently of the potential $q(x)$.

Theorem [6]. Let $\Omega \subset R^2$ be a smooth and bounded domain, and let Γ be a Lipschitz curve in $\bar{\Omega}$ which divides Ω into exactly two components. Then there exists for every given $\varepsilon > 0$ a potential $q_\varepsilon \in L^\infty(\Omega)$ such that $Z_2(q_\varepsilon) \subset \{x \in \Omega \; ; \; d(x,\Gamma) \leq \varepsilon\}$, where $Z_2(q_\varepsilon)$ denotes the nodal line of the second eigenfunction v_2 of (1) with $q = q_\varepsilon$.

It is easy to see that the Theorem implies the following

Corollary. There exists no upper bound to the length of the nodal line of the second eigenfunction uniformly for all potentials.

The idea of the proof of the theorem is the following:

Let $\Omega_n = \{ x \in \Omega \; ; \; d(x,\Gamma) \leq \frac{1}{n} \}$, and define (suitably) a sequence of potentials $q_n \geq 0$ such that $q_n(x) = r_n$ for $x \in \Omega_n$, and $q_n(x) \leq c$ for $x \in \Omega \setminus \Omega_n$.
Using the equation

$$(4) \qquad - \Delta v_{2,n} + q_n v_{2,n} = \lambda_{2,n} v_{2,n} ,$$

where $v_{2,n}$ denotes the second eigenfunction, one estimates

$$\|v_{2,n}\|^2_{H^1(\Omega_n)} \leq c_1 , \quad \text{for all} \quad n \in N ,$$

$$\|v_{2,n}\|^2_{L^2(\Omega_n)} \leq c_1/r_n , \quad \text{for all} \quad n \in N.$$

With this one now estimates the trace of $v_{2,n}$ on the boundary of Ω_n in $L^2(\partial\Omega_n)$:

$$\begin{aligned} \|v_{2,n}\|_{L^2(\partial\Omega_n)} &\leq c_n \|v_{2,n}\|_{H^{3/4}(\Omega_n)} \leq \\ &\leq c_n \, d_n \, \|v_{2,n}\|^{1/4}_{L^2(\Omega_n)} \|v_{2,n}\|^{3/4}_{H^1(\Omega_n)} \leq \\ &\leq c_n \, d_n \left(\frac{c_1^4}{r_n}\right)^{1/8} , \end{aligned}$$

where c_n denotes the embedding constant of $H^{3/4}(\Omega_n)$ into $L^2(\partial\Omega_n)$, and d_n the interpolation constant of $H^{3/4}(\Omega_n)$ between $L^2(\Omega_n)$ and $H^1(\Omega_n)$. Note that these estimates hold independently of r_n. Hence, choosing r_n such that $(c_n d_n)/r_n^{1/8} \to 0$ as $n \to \infty$ we see that $v_{2,n} \to 0$ as $n \to \infty$ in $L^2(\Gamma)$.

This allows to pass to the limit in equation (4), with the limiting equation

$$(5) \qquad \begin{aligned} -\Delta\overline{v}_2 + q\,\overline{v}_2 &= \lambda_2\,\overline{v}_2 \ , \quad \text{in} \quad \Omega \setminus \Gamma \\ \overline{v}_2 &= 0 \qquad \text{on} \quad \partial\Omega \cup \Gamma \ . \end{aligned}$$

Using the choice of $q_n(x)$ on $\Omega \setminus \Omega_n$ one now proves that λ_2 is equal to the first Dirichlet eigenvalue on the two subdomains, and that $\overline{v}_2 = \alpha w_1 + \beta z_1$, where w_1 and z_1 are the first eigenfunctions on the two subdomains (extended by zero to the other, respectively). Now, if $\alpha \neq 0$ and $\beta \neq 0$, then $\overline{v}_2 \neq 0$ on $\Omega \setminus \Gamma$, and then one proves easily that $Z_2(q_n) \subset \{ x \in \Omega \ ; \ d(x,\Gamma) \leq \varepsilon \}$ for $n \geq n_o$. In case that $\alpha = 0$ and $\beta \neq 0$ (or vice versa) one needs an additional argument: Setting $v_{2,n}^+ = \max\{v_{2,n}, 0\}$ and $f_n = v_{2,n}^+ / \|v_{2,n}^+\|_{L^2(\Omega)}$ one proves that $f_n \to w_1$ as $n \to \infty$ in $L^2(\Omega)$, where w_1 is again the first eigenfunction on one of the subdomains. This thenallows to conclude the proof.

In [7] a similar result is obtained for the equation

$$(6) \qquad \begin{aligned} -\Delta v &= \lambda\,\rho(x)\,v \ , \quad \text{in} \quad \Omega \subset R^2 \\ v &= 0 \ , \quad \text{on} \quad \partial\Omega \ , \end{aligned}$$

where $\rho : \Omega \to R$. In the proof a sequence ρ_n is constructed which tends to minus infinity in a strip $\{ x \in \Omega; \ d(x,\Gamma) \leq 1/n \}$ around Γ.

Problems of the form (6) with functions ρ which assume negative values occur as linearizations of certain nonlinear problems.

If the function ρ in (6) is positive-valued, then equation (6) describes the stationary solutions of a non-homogeneous membrane. It is not known if a result similar to the one stated holds for the class of positive functions ρ. In particular, it is not known if the length of the nodal line of the second eigenfunction is uniformly bounded independently of $\rho > 0$.

The mentioned results seem to rule out that the length of the nodal line plays a role in a geometric characterization of eigenfunctions. Such a characterization would be highly important in applications to nonlinear differential equations. In fact, in one dimension many important results in bifurcation theory and variational methods are due to the stated nodal-point characterization.

Of course one can think of other ways to generalize the one-dimensional theorem. For instance, one could count critical points, turning points, etc., of the eigenfunction to obtain properties of the eigenfunctions which are invariant under the change of the potential q(x). However, there are no successful attempts in these directions up to now.

References.

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