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ON SOME BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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The theory of boundary value problems for systems of ordinary differential equations has been as a matter of fact constructed in the last thirty years. During that time the a priori estimate techniques and topological methods were essentially developed, enabling one to establish the solvability and correctness for a wide class of nonlinear boundary value problems (see [1] and references cited therein).

The present work contains new - not included in [1] - sufficient conditions of the solution existence for a system of vector differential equations

$$\frac{du_i}{dt} = f_i(t, u_1, \dots, u_m) \quad (i = 1, \dots, m) \tag{1}$$

satisfying boundary conditions of the form

$$\ell(u_1, \dots, u_m) = h(u_1, \dots, u_m). \tag{2}$$

These results were obtained by the method having much in common with A.M. Liapunov's second method.

The following notation is used in the paper:

$$R =]-\infty, +\infty[\,, \quad R_+ = [0, +\infty[\,;$$

R^k is a k-dimensional real Euclidean space of vectors $x = (\xi_i)_{1 \le i \le k}$ with the norm

$$\|x\| = \sum_{i=1}^{k} |\xi_i| \,;$$

$$R_+^k = \{(\xi_i)_{1\le i\le k} \in R^k : \xi_1 \ge 0, \dots, \xi_k \ge 0\};$$

$R^{k\times n}$ is the space of real $k\times n$ -matrices $X = (\xi_{ij})_{\substack{1\le i\le k\\ 1\le j\le n}}$ with the norm

$$\|X\| = \sum_{i=1}^{k}\sum_{j=1}^{n} |\xi_{ij}|;$$

$x\cdot y$ is the scalar product of the vectors x and $y \in R^k$;

Ax is the product of the matrix $A \in R^{k\times n}$ and the column-vector $x \in R^n$;

$\det A$ is the determinant of the matrix A, A^* is the transposed matrix A;

$C([a,b];R^n)$ and $C([a,b];R^{n\times n})$ are the spaces of continuous vector functions $x:[a,b]\to R^n$ and of matrix functions $X:[a,b]\to R^{n\times n}$;

$$\|x\|_C = \max\{\|x(t)\| : a\le t\le b\};$$

$C(R^n;R)$ is the space of continuous functions $W: R^n \to R$, while $C^1(R^n;R)$ is the space of functions $W: R^n\to R$ having continuous partial derivatives of the first order;

$L([a,b];R)$ and $L^p([a,b];R^{n\times n})$ are, respectively, the space of summable functions $x:[a,b]\to R$ and the space of matrix functions $X:[a,b]\to R^{n\times n}$ with components summable to the power p;

$$C(R^n;R_+) = \{W\in C(R^n;R) : W(x)\ge 0 \text{ for } x\in R^n\};$$

$$L([a,b];R_+) = \{y\in L([a,b];R) : y(t)\ge 0 \text{ for } t\in[a,b]\};$$

$K([a,b]\times\mathcal{D}_1;\mathcal{D}_2)$ is the Caratheodory class, i.e. the set mappings $f:[a,b]\times\mathcal{D}_1\longrightarrow\mathcal{D}_2$ such that $f(\cdot,x):[a,b]\longrightarrow\mathcal{D}_2$ is measurable for any $x\in\mathcal{D}_1$, $f(t,\cdot):\mathcal{D}_1\longrightarrow\mathcal{D}_2$ is continuous for almost all $t\in[a,b]$ and

$$\sup\{\|f(\cdot,x)\|: x\in\mathcal{D}_0\}\in L([a,b];R_+)$$

for any compactum $\mathcal{D}_0\subset\mathcal{D}_1$.

In what follows it will be assumed everywhere that

$$f_i\in K([a,b]\times R^n;R^{n_i})\ (i=1,\dots,m),\ n=\sum_{i=1}^{m}n_i,$$

$\ell:C([a,b];R^n)\longrightarrow R^n$ is a linear continuous operator $h:C([a,b];R^n)\longrightarrow R^n$ is a nonlinear continuous operator bounded on each bounded set of the space $C([a,b];R^n)$.

Prior to proceeding to the formulation of the main results let us introduce

D e f i n i t i o n. Let $W\in C^1(R^n;R)$. The operators $\ell:C([a,b];R^n)\to R^n$ and $h:C([a,b];R^n)\longrightarrow R^n$ will be said to be W-compatible if there exist a matrix function $\mathcal{P}\in L([a,b];R^{n\times n})$ and a positive number γ such that

1) the homogeneous problem

$$\frac{du}{dt}=\mathcal{P}(t)u,\quad \ell(u)=0$$

has the zero solution only;

2) the inequality

$$\mathcal{P}(t)x\cdot \operatorname{grad}W(x)\geq 0$$

is fulfilled on $[a,b]\times R^n$;

3) the estimate

$$W(u(b))-W(u(a))\leq\gamma$$

holds for any vector function $u\in C([a,b];R^n)$ satisfying the equality $\ell(u)=\alpha h(u)$ for some $\alpha\in[0,1]$.

T h e o r e m 1. Let there exist a function $W\in C^1(R^n;R)$ such that the operators ℓ and h are W-compatible and on $[a,b]\times R^n$ we have the inequalities

$$f_0(t,x_1,\dots,x_m) \geq \delta(t,\|x_1\|,\dots,\|x_m\|) - \delta_1(t) \qquad (3)$$

and

$$\|f_1(t,x_1,\dots,x_m)\| \leq [\delta_1(t)+\gamma_1|f_0(t,x_1,\dots,x_m)|](1+\|x_1\|),$$
$$\|f_i(t,x_1,\dots,x_m)\| \leq$$
$$\leq [\delta_i(t,x_1,\dots,x_{i-1})+\gamma_i(x_1,\dots,x_{i-1})|f_0(t,x_1,\dots,x_m)|](1+\|x_i\|) \quad (i=2,\dots,m) \qquad (4)$$

where

$$f_0(t,x_1,\dots,x_m) = \sum_{i=1}^{m} f_i(t,x_1,\dots,x_m)\cdot \mathrm{grad}_{x_i} W(x_1,\dots,x_m),$$

$\delta_1 \in L([a,b];R_+)$, $\gamma_1 \in R_+$, $\delta_i \in K([a,b]\times R^{n_1+\dots+n_{i-1}};R_+)$, $\gamma_i \in C(R^{n_1+\dots+n_{i-1}};R_+)$ $(i=2,\dots,m)$ and the function $\delta:[a,b]\times R_+^m \to R_+$ is summable with respect to the first variable, does not decrease with respect to the last m variables and

$$\lim_{\sum_{i=1}^{m}\rho_i \to +\infty} \int_a^b \delta(t,\rho_1,\dots,\rho_m)\,dt = +\infty . \qquad (5)$$

Then Problem (1),(2) is solvable.

R e m a r k 1. Condition (5) is essential and cannot be neglected unless additional restrictions are imposed on the operators ℓ and h .

R e m a r k 2. In the right-hand side of each of inequalities (4) the multiplier $1+\|x_i\|$ cannot be replaced by the multiplier $(1+\|x_i\|)^{1+\varepsilon}$ no matter how small $\varepsilon > 0$ is.

Let us consider the case when the boundary conditions (2) have the form

$$u_i(a)=B_i u_i(b)+$$
$$+\varphi_i(u_1(a),\dots,u_m(a),u_1(b),\dots,u_m(b)) \ (i=1,\dots,m) \qquad (2^{\mathrm{I}})$$

where

$$B_i\in R^{n_i\times n_i}\ ,\ \varphi_i\in C(R^{2n};R^{n_i})\ (i=1,\dots,m).$$

From Theorem 1 we obtain

C o r o l l a r y 1. Let there exist nonsingular symmetric matrices $A_i\in R^{n_i\times n_i}$ $(i=1,\dots,m)$ such that on $[a,b]\times R^n$ we have inequalities (3) and (4) where

$$f_0(t,x_1,\dots,x_m)=\sum_{i=1}^{m} f_i(t,x_1,\dots,x_m)\cdot A_i x_i\ ,$$

γ_1 , δ_1 , γ_i , δ_i $(i=2,\dots,m)$ and δ are a number and functions satisfying the conditions of Theorem 2. Let, besides, one of the conditions below be fulfilled for any $i\in\{1,\dots,m\}$:

1) the matrix $B_i^* A_i B_i - A_i$ is positively defined and the function φ_i is bounded;

2) the matrix $B_i^* A_i B_i - A_i$ is non-negatively defined and the function $(x_1,\dots,x_m,y_1,\dots,y_m)\to(\|x_i\|+\|y_i\|)\varphi_i(x_1,\dots,x_m,y_1,\dots,y_m)$ is bounded.

Then Problem (1),(2^{I}) is solvable.

Let us concretize Theorem 1 for the boundary value problems

$$\frac{du_1}{dt}=f_1(t,u_1,u_2)\ ,\ \frac{du_2}{dt}=f_2(t,u_1,u_2), \qquad (6)$$

$$u_{i_1}(a)=\varphi_1(u_1(a),u_2(a),u_1(b),u_2(b)),$$
$$u_{i_2}(b)=\varphi_2(u_1(a),u_2(a),u_1(b),u_2(b)) \qquad (7)$$

and

$$u'' = f(t, u, u'), \tag{8}$$

$$\begin{aligned} u^{(i_1-1)}(a) &= \Psi_1(u(a), u'(a), u(b), u'(b)), \\ u^{(i_2-1)}(b) &= \Psi_2(u(a), u'(a), u(b), u'(b)) \end{aligned} \tag{9}$$

where i_1 and $i_2 \in \{1, 2\}$,

$$f_i \in K([a,b] \times R^{n_1+n_2}; R^{n_i}),\ \varphi_i \in C(R^{2n_1+2n_2}; R^{n_i}),$$

$$f \in K([a,b] \times R^{2n}; R^n),\ \Psi_i \in C(R^{4n}; R^n).$$

T h e o r e m 2. Let there exist a natural number n_0 and matrices $A_i \in R^{n_0 \times n_i}$ $(i=1,2)$ such that on $[a, b] \times R^{n_1+n_2}$ we have the inequalities

$$\begin{aligned} &f_0(t, x_1, x_2) \geq \delta(t, \|x_1\|, \|x_2\|) - \delta_1(t), \\ &\|f_1(t, x_1, x_2)\| \leq [\delta_1(t) + \gamma_1 |f_0(t, x_1, x_2)|](1 + \|x_1\|), \\ &\|f_2(t, x_1, x_2)\| \leq [\delta_2(t, x_1) + \gamma_2(x_1) |f_0(t, x_1, x_2)|](1 + \|x_2\|) \end{aligned} \tag{10}$$

and on $R^{2n_1+2n_2}$ the inequality

$$A_{i_2}\varphi_2(x_1, x_2, y_1, y_2) \cdot A_{3-i_2} y_{3-i_2} - A_{i_1}\varphi_1(x_1, x_2, y_1, y_2) \cdot A_{3-i_1} x_{3-i_1} \leq \\ \leq \gamma_1$$

where

$$f_0(t,x_1,x_2)=A_1 f_1(t,x_1,x_2)\cdot A_2 x_2 + A_1 x_1 \cdot A_2 f_2(t,x_1,x_2), \tag{11}$$

$\delta_1 \in L([a,b];R_+)$, $\gamma_1 \in R_+$, $\delta_2 \in K([a,b]\times R^{n_1};R_+)$, $\gamma_2 \in C(R^{n_1};R_+)$, and the function $\delta:[a,b]\times R_+^2 \to R_+$ is summable with respect to the first variable, does not decrease with respect to the last two variables and

$$\lim_{\rho_1+\rho_2 \to +\infty} \int_a^b \delta(t,\rho_1,\rho_2)\,dt = +\infty . \tag{12}$$

Then Problem (6),(7) is solvable.

R e m a r k 3. If i_1 and φ_1 is bounded, then (10) and (12) can be replaced by the conditions

$$f_0(t,x_1,x_2) \geq \delta(t,\|x_2\|) - \delta_1(t)$$

and

$$\lim_{\rho \to +\infty} \int_a^b \delta(t,\rho)\,dt = +\infty ; \tag{13}$$

if however $i_1=1$, $i_2=2$, φ_1 is bounded and

$$\|\varphi_2(x_1,x_2,y_1,y_2)\| \leq \varphi_0(x_1,y_1)$$

where $\varphi_0 : R^{2n_1} \to R_+$ is a continuous function, then instead of (10) and (12) it can be assumed that

$$f_0(t,x_1,x_2) \geq -\delta_1(t) .$$

C o r o l l a r y 2. Let there exist a positively defined symmetric matrix $A \in R^{n\times n}$ such that on $[a,b]\times R^{2n}$ we have the inequalities

$$f(t,x,y)\cdot Ax \geq \delta(t,\|x\|)-\delta_0(t),$$

$$\|f(t,x,y)\| \leq [\delta_1(t,x)+\gamma_1(x)\|y\|^2](1+\|y\|),$$

and on R^{4n} the inequality

$$\Psi_2(x_1,x_2,y_1,y_2)\cdot Ay_{3-i_2}-\Psi_1(x_1,x_2,y_1,y_2)\cdot Ax_{3-i_1} \leq \gamma \qquad (14)$$

where $\delta_0 \in L([a,b];R_+)$, $\delta_1 \in K([a,b]\times R^n;R_+)$, $\gamma_1 \in C(R^n;R_+)$, $\gamma \in R_+$, and the function $\delta:[a,b]\times R_+ \to R_+$ is summable with respect to the first variable, does not decrease with respect to the second variable and satisfies condition (13). Then Problem (8), (9) is solvable.

R e m a r k 4. Let

$$\Psi_k(x_1,x_2,y_1,y_2)=B_{k1}x_{3-i_1}+B_{k2}y_{3-i_2} \quad (k=1,2).$$

Then condition (14) is fulfilled if and only if the matrix

$$\begin{pmatrix} AB_{11} & AB_{12} \\ -AB_{21} & -AB_{22} \end{pmatrix}$$

is non-negatively defined.

Finally, we shall give one more existence theorem for Problem (6),(7), complementing the above results.

T h e o r e m 3. Let $i_1=1$, $i_2=2$, the function φ_1 be bounded and the function φ_2 admit the estimate

$$\|\varphi_2(x_1,x_2,y_1,y_2)\| \leq \varphi_0(x_1,x_2)$$

where $\varphi_0 \in C(R^{2n_1};R_+)$. Let furthermore there exist a natural number n_0 and matrices $A_i \in R^{n_0\times n_i}$ $(i=1,2)$ such that on $[a,b]\times R^{n_1+n_2}$ we have the inequalities

$$f_0(t,x_1,x_2) \geq -\delta_0(t,\|x_1\|+\|x_2\|),$$

$$\|f_1(t,x_1,x_2)\| \leq \delta_1(t)(1+\|x_1\|)+|\delta_1(t)f_0(t,x_1,x_2)|^{1/2},$$

$$\|f_2(t,x_1,x_2)\| \leq \delta_2(t,x_1)(1+\|x_2\|)+|\delta_2(t,x_2)f_0(t,x_1,x_2)|^{1/2}$$

and on $R^{2n_1+2n_2}$ the inequality

$$A_2\varphi_2(x_1,x_2,y_1,y_2)\cdot A_1y_1 - A_1\varphi_1(x_1,x_2,y_1,y_2)\cdot A_2x_2 \leq$$
$$\leq \gamma(\|x_1\|+\|x_2\|),$$

where f_0 is the function given by (11), $\delta_1 \in L([a,b];R_+)$, $\delta_2 \in K([a,b]\times R^{n_1};R_+)$, $\gamma: R_+ \to R_+$ is a nondecreasing function satisfying the condition

$$\lim_{\rho\to+\infty}\frac{\gamma(\rho)}{\rho^2}=0$$

and the function $\delta_0:[a,b]\times R_+ \to R_+$ is summable with respect to the first variable, does not decrease with respect to the second variable and

$$\lim_{\rho\to+\infty}\frac{1}{\rho^2}\int_a^b \delta_0(t,\rho)\,dt = 0 \, .$$

Then Problem (6),(7) is solvable.

As an example we shall consider the following boundary value problem that arises in the optimal control theory (see 2 , § 3.2):

$$\frac{du_1}{dt}=F(t)u_1+H(t)g_1(t)H^*(t)u_2,$$
$$\frac{du_2}{dt}=-F^*(t)u_1+g_2(t)u_2\,, \qquad (15)$$

$$u_1(a) = \varphi_1(u_2(a)), \; u_2(b) = \varphi_2(u_1(b)) \tag{16}$$

where

$$F \in L([a,b]; R^{n\times n}), \quad H \in L^2([a,b]; R^{n\times n}),$$

$$g_1 \in C([a,b]; R^{n\times n}), \; g_2 \in L([a,b]; R^{n\times n})$$

$$\varphi_i \in C(R^n; R^n) \quad (i=1,2).$$

From Theorem 3 we obtain

C o r o l l a r y 3. Let the matrix $g_1(t)$ be positively defined for any $t \in [a,b]$ and let the matrix $g_2(t)$ be non-negatively defined for almost all $t \in [a,b]$. Let furthermore the inequa-

$$\varphi_1(x)\cdot x \geq 0, \quad (\varphi_2(x) - \varphi_2(0))\cdot x \leq 0$$

be fulfilled on R^n. Then Problem (15), (16) is solvable.

R e f e r e n c e s

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