SOME PROBLEMS OF THE THEORY OF SOBOLEV SPACES OF INFINITE ORDER AND OF NONLINEAR EQUATIONS

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The Sobolev spaces of infinite order arise in a natural way as energetic spaces corresponding to the Dirichlet problem for nonlinear elliptic equations of infinite order. For example, the Sobolev space of infinite order

$$\overset{\circ}{W}{}^{\infty}\{a_\alpha,p_\alpha\} = \{u(x) \in C_0^\infty(G) : \rho(u) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{p_\alpha}^{p_\alpha} < \infty\},$$

where $a_\alpha \geqq 0$, $p_\alpha \geqq 1$ are number sequences, $\|\cdot\|_r$ is the norm in L_r and $C_0^\infty(G) = \{u(x) \in C^\infty(G): D^\omega u|_\Gamma = 0 \ , \ |\omega| = 0,1,\ldots\}$ corresponds to the problem

$$(0.1) \quad L(u) \equiv \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha(a_\alpha |D^\alpha u|^{p_\alpha-2} D^\alpha u) = h(x), \quad x \in G ,$$

$$(0.2) \quad D^\alpha u|_\Gamma = 0 \ , \ |\omega| = 0,1,\ldots \ .$$

The metric $\rho(u)$ is a natural metric for the problem (0.1), (0.2). It is evident that even conversely, when studying the operator $L(u)$ in the space $\overset{\circ}{W}{}^{\infty}\{a_\alpha,p_\alpha\}$, it is possible to consider the problem of solvability of (0.1), (0.2) in this space. However, it should be pointed out that before doing so it is necessary to study the spaces $\overset{\circ}{W}{}^{\infty}\{a_\alpha,p_\alpha\}$ themselves and, above all, to solve the problem of their existence, i.e. the problem of their non-triviality. Besides it is the opinion of the author that problems such as the theory of traces in spaces of infinite order imbedding theorems, problems of the geometry, of density e.t.c. are of interest by themselves. Not all of these problems have been solved until now with sufficient profundity. Some of them will be dealt with in the present lecture. Namely, we shall study the following problems :

1) non-triviality of Sobolev spaces of infinite order;

2) theory of traces;

3) imbedding theorems.

Simultaneously we shall consider some problems of the theory of nonlinear equations :

1) solvability of the Dirichlet problem of infinite order (the Cauchy-Dirichlet problem);

2) behaviour of solutions of nonlinear elliptic equations of an order $2m$ when $m \to +\infty$;

3) non-homogeneous Cauchy-Dirichlet problem a. o.

Let us proceed to explaining exactly our results.

1. Non-triviality of Sobolev spaces of infinite order

A criterion of non-triviality of the spaces $W^{\infty}\{a_\alpha, p_\alpha\}$ depends essentially on the definition domain of functions $u(x)$ which are elements of the space. Three cases which are the most characteristic ones for the analysis will be studied here : $G \subset \mathbb{R}^\nu$ - a bounded domain, $G = \mathbb{R}^\nu$ and $G = S^1 \times \ldots \times S^1$ - a torus.

a) (the case of a bounded domain G). Let $a_\alpha \geqq 0$, $p_\alpha \geqq 0$ and $r_\alpha \geqq 1$ be arbitrary number sequences, $\alpha = (\alpha_1, \ldots, \alpha_\nu)$, $|\alpha| = \alpha_1 + \ldots + \alpha_\nu$. Consider the space

$$\overset{\circ}{W}{}^{\infty}\{a_\alpha, p_\alpha\} = \{u(x) \in C_0^\infty(G) : \rho(u) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \| D^\alpha u \|_{r_\alpha}^{p_\alpha} < \infty\}.$$

<u>Definition 1.1</u>. *The space* $\overset{\circ}{W}{}^{\infty}\{a_\alpha, p_\alpha\}$ *is called non-trivial if it contains at least one function not identically equal to zero.*

Before formulating a non-triviality criterion, let us introduce the following number sequence : Let M_N, $N = 0,1,\ldots$ be the solutions of the equations

$$\sum_{|\alpha|=N} a_\alpha M_N^{p_\alpha} = 1$$

with $M_N = +\infty$ if $a_\alpha = 0$ for all $|\alpha| = N$.

THEOREM 1.1. *The space* $\overset{o}{W}{}^\infty\{a_\alpha, p_\alpha\}$ *is non-trivial if and only if the sequence* M_N, $N = 0,1,\ldots$ *defines a non-quasianalytical Hadamard's class of one real variable.*

Making use of the criterion of non-analyticity due to Mandelbrojt-Bang we can formulate Theorem 1.1 as follows :

The space $\overset{o}{W}{}^\infty\{a_\alpha, p_\alpha\}$ *is non-trivial if and only if*

$$\lim_{N\to\infty} M_N^{\frac{1}{N}} = \infty \ , \qquad \sum_{N=0}^{\infty} \frac{M_N^C}{M_{N+1}^C} < \infty \quad ,$$

where M_N^C *is the convex regularization of the sequence* M_N *by means of logarithms (see* [1] *).*

REMARK. The numbers $r_\alpha \geqq 1$ are not included in the notation of the space $\overset{o}{W}{}^\infty\{a_\alpha, p_\alpha\}$ due to the fact that they play no role in the problems of non-triviality.

Example. Let $G = (a,b)$, $a_n = (n!)^{-q}$, $q > 1$. The corresponding space $\overset{o}{W}{}^\infty\{a_n, p\}$ is non-trivial provided $p < q$.

b) (the case $G = \mathbb{R}^\nu$). Let

$$W^\infty\{a_\alpha, p_\alpha\} = \{u(x) \in C^\infty(\mathbb{R}^\nu) \ : \ \rho(u) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \| D^\alpha u \|_{r_\alpha}^{p_\alpha} < \infty\}.$$

THEOREM 1.2. *The space* $W^\infty\{a_\alpha, p_\alpha\}$ *is non-trivial if and only if there exists* $q = (q_1, \ldots, q_\nu), q_j > 0$ *such that*

$$\sum_{|\alpha|=0}^{\infty} a_\alpha q^{\alpha p_\alpha} \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha q_1^{\alpha_1 p_\alpha} \ldots q_\nu^{\alpha_\nu p_\alpha} < \infty \ .$$

Example. Let $p_\alpha = p$, $|\alpha| = 0,1,\ldots$. Then the non-triviality condition for $W^\infty\{a_\alpha, p\}$ means that the function

$$a(z) = \sum_{N=0}^{\infty} b_N z^N \quad (b_N = \sum_{|\alpha|=N} a_\alpha)$$

of a complex variable $z \in \mathbb{C}^1$ is analytic at zero.

c) (the case $G = T^\nu$, where T^ν is a torus of dimension ν)

Let

$$W^\infty\{a_\alpha, p_\alpha\} = \{u(x) \in C^\infty(T^\nu) : \rho(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha u||_{r_\alpha}^{p_\alpha} < \infty \}$$

be the Sobolev space of infinite order on a torus.

THEOREM 1.3. *The space* $W^\infty\{a_\alpha, p_\alpha\}$ *is non-trivial and of infinite dimension if and only if there exists a sequence of mutually different non-negative multiindices* $q_n = (q_{1n}, \ldots, q_{\nu n})$, $n = 0,1,\ldots$ *such that*

$$\sum_{|\alpha|=0}^{\infty} a_\alpha q_n^{\alpha p_\alpha} < \infty \quad .$$

<u>Example</u>. Let $p_\alpha \equiv p$, $|\alpha| = 0,1,\ldots$. Then the space $W^\infty\{a_\alpha, p\}$ is non-trivial and of infinite dimension if and only if the function

$$a(z) \equiv \sum_{N=0}^{\infty} b_N z^N \quad (b_N = \sum_{|\alpha|=N} a_\alpha), \quad z \in C^1$$

is an entire function.

2. Theory of traces

Let $G \in \mathbb{R}^\nu$ is a domain with a boundary Γ . Consider the space

$$W^\infty\{a_\alpha, p_\alpha\} = \{u(x) \in C^\infty(G) : \rho(u) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha u||_{p_\alpha}^{p_\alpha} < \infty \}.$$

We shall assume that the corresponding space $\overset{\circ}{W}{}^\infty\{a_\alpha, p_\alpha\}$ is non-trivial.

Further, let a family of functions be given on Γ : $f_\omega(x')$, $x' \in \Gamma$, $|\omega| = 0,1,\ldots$. We say that this family is a trace in the space $W^\infty\{a_\alpha, p_\alpha\}$ if there exists a function $u(x) \in W^\infty\{a_\alpha, p_\alpha\}$ such that

$$D^\omega u|_\Gamma = f_\omega(x') \quad , \quad |\omega| = 0,1,\ldots \quad .$$

We need the following notation when formulating the trace criterion. Let us assume that for $N = 1,2,\ldots$ the family $f_\omega(x')$,

$|\omega| \leqq N - 1$ has an extension in the space

$$W^N\{a_\alpha,\ p_\alpha\} \equiv \{u(x):\ \rho_N(u) \equiv \sum_{|\alpha|=0}^{N} a_\alpha ||D^\alpha u||_{p_\alpha}^{p_\alpha} < \infty\}$$

Let E_N be the family of all such extensions and let us denote by $u_N(x)$ that one for which

$$\rho_N(u_N) = \inf \rho_N(u)\ ,\quad u \in E_N\ .$$

THEOREM 2.1. *A family of boundary functions* $f_\omega(x')$, $|\omega| = 0,1,\ldots,$ *is a trace in* $W^\infty\{a_\alpha,p_\alpha\}$ *if and only if it satisfies the following conditions :*

(i) for each $N = 1,2,\ldots$ *the family* $f_\omega(x')$, $|\omega| \leqq N - 1$ *has an extension in* $W^N\{a_\alpha,p_\alpha\}$ *;*

(ii) there is a constant $K > 0$ *such that* $\rho_N(u_N) \leqq K$ *for all* N .

REMARK. If $p_\alpha > 1$ then $u_N(x)$ is obviously a solution of the nonhomogeneous boundary value problem

$$(*)\qquad \begin{gathered} \sum_{|\alpha|=0}^{N} (-1)^{|\alpha|} D^\alpha (a_\alpha |D^\alpha u|^{p_\alpha - 2} D^\alpha u) = 0\ , \\ D^\omega u|_\Gamma = f_\omega(x')\ ,\quad |\omega| \leqq N - 1\ , \end{gathered}$$

and consequently, the above criterion is effective to the same degree as the methods of estimation of solutions of (*) . In particular, for $p_\alpha \equiv 2$ the problem reduces to uniform estimates of the Green functions provided $N \to \infty$. There are no theories of such estimates at present, hence it is essential to give sufficient, nonetheless easily verifiable trace conditions.

Sufficient conditions for a trace in a strip.

Let $G = [0,a] \times \mathbb{R}^\nu$ be a strip of variables t, x . Consider the space

$$W^\infty\{a_{n\alpha},p\} = \{u(t,x)\ :\ \rho(u) \equiv \sum_{n+|\alpha|=0}^{\infty} a_{n\alpha} ||D_t^n D_x^\alpha u||_p^p < \infty\}\ .$$

Our aim is to formulate assumptions on functions $\phi_m(x)$, $m = 0,1,\ldots$ under which there exists a function $u(t,x) \in W^\infty\{a_{n\alpha},p\}$ satisfying the conditions

$$D_t^m u(0,x) = \phi_m(x), \quad D_t^m u(a,x) = 0, \quad x \in \mathbb{R}^\nu, \quad m = 0,1,\ldots$$

(the case of non-zero conditions at $t = a$ is dealt with analogously). However, it is necessary above all to establish non-triviality of the space

$$\overset{o}{W}{}^\infty\{a_{n\alpha},p\} = \{u(t,x) : \rho(u) < \infty, \ D^m u|_{t=a,0} = 0, \ m = 0,1,\ldots\} .$$

THEOREM 2.2. *The space* $\overset{o}{W}{}^\infty\{a_{n\alpha},p\}$ *is non-trivial if and only if the following conditions are satisfied :*

(i) there is such a $q = (q_1,\ldots,q_\nu)$, $q_\nu > 0$, *that*

$$b_n \overset{\text{def}}{=} \sum_{|\alpha|=0}^{\infty} a_{n\alpha} q^{\alpha p} < \infty$$

for all $n = 0,1,\ldots$;

(ii) the sequence $M_n = \{b_n^{-1/p}$ *if* $b_n > 0$, $+\infty$ *if* $b_n = 0\}$ *defines a non-quasianalytical Hadamard's class.*

In order to formulate sufficient trace conditions we introduce a number sequence

$$S_m = \sum_{k=0}^{\infty} M_{m+k}^C / M_{m+k+1}^C$$

where M_n^C is a convex regularization by means of logarithms of the sequence M_n . Let us note that $S_m \to 0$ with $m \to \infty$ in virtue of the condition (ii) of Theorem 2.2.

THEOREM 2.3. *Let the following conditions be fulfilled :*

(i) the numbers

$$\rho_m = \sup_n (M_n^C \sum_{|\alpha|=0}^{\infty} a_{n\alpha}^{1/p} ||D^\alpha \phi_m(x)||_{L_p(\mathbb{R}^\nu)})$$

are finite for all $m = 0,1,\ldots$;

(ii) there is a number $r > 0$ *such that*

$$\sum_{m=0}^{\infty} \rho_m \max\left[\frac{S_m r^m}{m!}, \ (M_m^C)^{-1}\right] < \infty .$$

Then there exists a function $u(t,x) \in W^\infty\{a_{n\alpha},p\}$ *which satisfies the conditions* (2.1).

The theorem is proved by constructing the desired function. Let us describe the crucial point of this construction. To this end, let $b < r/2$, $d_m = 4S_m/b$ and $b_{km} = M_m^C/d_m^k M_{m+k}^C$, $m=0,1,\dots$. Evidently

$$b_{0m}/b_{1m} + b_{1m}/b_{2m} + \dots < b/3$$

and consequently, there exists a family of "canonical" functions $\Phi_m(t) \in C^\infty(0,b)$ satisfying the inequality

$$|D^k\Phi_m(t)| \leq q^k d_m^k M_{m+k}^C/M_m^C \ , \quad k = 0,1,\dots \ , \quad t \in [0,b]$$

with $q < 1$ (see [1], [2]). Now, let us put

$$v(t,x) = \sum_{m=0}^{\infty} v_m(t)\phi_m(x) \ ,$$

where

$$v_0(t) = \Phi_0(t), \quad v_1(t) = d_1 \int_0^{t/d_1} \Phi_1(\eta)d\eta \ ,$$

$$v_m(t) = \frac{1}{(m-2)!}\int_0^t (t-\tau)^{m-2}\int_0^{\tau/d_m} \Phi_m(\eta)d\eta d\tau \ , \ m = 2,3,\dots \ .$$

By the condition (ii) of Theorem 2.2 we conclude $\rho(v) < \infty$.

<u>Example</u>. Let $a_{n\alpha} = a_n \cdot a_\alpha$ where a_n^{-1} is a logarithmically convex sequence. The space $\overset{\circ}{W}{}^\infty\{a_n \cdot a_\alpha, p\}$ is non-trivial if and only if the same holds for the spaces $\overset{\circ}{W}{}^\infty\{a_n,p\}(0, a)$ and $W^\infty\{a_\alpha,p\}(\mathbb{R}^\nu)$. It is evident that in this case we may put $b_n = a_n$. Then obviously

$$\rho_m = \sum_{|\alpha|=0}^{\infty} a_\alpha^{1/p}||D^\alpha\phi_m(x)||_{L_p(R^\nu)}$$

and the condition (ii) assumes the form

$$\sum_{m=0}^{\infty} \rho_m \max\left(\frac{S_m r^m}{m!}, a_m^{1/p}\right) < \infty \tag{2.2}$$

with $S_m = (a_{m+1}/a_m)^{1/p} + (a_{m+2}/a_{m+1})^{1/p} + \dots$. In particular, if $a_{n+1} \leq a_n^2$, $a_0 < 1$, then the condition (2.2) assumes the form

$$\sum_{m=0}^{\infty} \rho_m a_m^{1/p} < \infty .$$

3. Imbedding theorems

Given two spaces $W^{\infty}\{a_\alpha,p\}$ and $W^{\infty}\{b_\alpha,q\}$ (the domain of definition of G being any one of those mentioned above) with norms

$$||u||_{a,\infty} = (\sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha u||_p^p)^{1/p}, \quad ||u||_{b,\infty} = (\sum_{|\alpha|=0}^{\infty} b_\alpha ||D^\alpha u||_q^q)^{1/q}$$

let

$$i_{\infty,\infty} : W^{\infty}\{a_\alpha,p\} \to W^{\infty}\{b_\alpha,q\} \tag{3.1}$$

be imbedding operator of the space $W^{\infty}\{a_\alpha,p\}$ into $W^{\infty}\{b_\alpha,q\}$.

Let m be a positive integer. Obviously there exists a number $r(m)$ such that for all $r \geq r(m)$ we have imbeddings

$$i_{r,m} : W^{r}\{a_\alpha,p\} \to W^{m}\{b_\alpha,q\} .$$

THEOREM 3.1. *The space* $W^{\infty}\{a_\alpha,p\}$ *is imbedded into the space* $W^{\infty}\{b_\alpha,q\}$ *if and only if the limit* $\lim\limits_{m\to\infty}\lim\limits_{r\to\infty} ||i_{r,m}||$ *exists and is finite. Moreover,*

$$||i_{\infty,\infty}|| = \lim_{m\to\infty}\lim_{r\to\infty} ||i_{r,m}||$$

holds.

The uniform boundedness of the norms $||i_{r,m}||$ *in the course of the limit process* $m \to \infty$ *represents a sufficient condition of the imbedding* (3.1).

In this way, in order to deal with individual imbedding it is necessary to have uniform estimates of constants in the inequalities

$$||u||_{b,m} \leq K_{r,m}||u||_{a,r}$$

where $||u||_{b,m}$ and $||u||_{a,r}$ are norms in $W^{m}\{b_\alpha,q\}$ and in $W^{r}\{a_\alpha,p\}$, respectively. Let us give some examples.

a) Let $G = \mathbb{R}^\nu$, $p = q = 2$. Then the answer can be given in terms of "characteristic" functions

$$a(\xi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \xi^\alpha \quad , \quad b(\xi) = \sum_{|\alpha|=0}^{\infty} b_\alpha \xi^\alpha \quad .$$

THEOREM 3.2. *Let* $a(\xi)$, $b(\xi)$ *be entire functions. The space* $W^\infty\{a_\alpha,2\}$ *is imbedded into* $W^\infty\{b_\alpha,2\}$ *if and only if there exists a constant* $K > 0$ *such that*

$$b(\xi) \leq Ka(\xi) \quad , \qquad \xi \in \mathbb{R}^\nu .$$

b) Let $q = p > 1$ *and* $\nu = 1$ *. We denote by* R_a *and* R_b *the radius of convergence of the series* $a(\xi)$ *and* $b(\xi)$ *, respectively.*

THEOREM 3.3. *Let the following conditions be fulfilled :*

(i) $\sum_{n=0}^{\infty} b_n R_a^n \left(\sum_{k=0}^{n} a_k R_a^k\right)^{-1} < \infty$, *if* $R_a < \infty$;

(ii) $\sum_{n=0}^{\infty} b_n \max_{\xi>0} (\xi^n a^{-1}(\xi)) < \infty$ *if* $R_a = \infty$.

Then $W^\infty\{a_n,p\} \subset W^\infty\{b_n,p\}$.

COROLLARY. If $R_a < R_b$, then the condition (i) is satisfied and, consequently, the imbedding from Theorem 3.3 takes place.

4. Cauchy-Dirichlet problem for nonlinear elliptic equations of infinite order

In a domain G let us consider the Cauchy-Dirichlet problem for a nonlinear elliptic equation of infinite order

$$L(u) \equiv \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x), \quad |\gamma| \leq |\alpha| \tag{4.1}$$

$$D^\omega u|_\Gamma = 0, \quad |\omega| = 0,1,\dots . \tag{4.2}$$

Here $A_\alpha(x,\xi_\gamma)$ are continuous functions of the variables $x \in G$ and all ξ_γ, $|\gamma| \leq |\alpha|$. We shall assume that the following conditions are fulfilled :

a) for any $x \in G$, ξ_γ and η_α with $|\alpha| = m$, $|\gamma| \leq m$ $(m \geq 0)$ the inequalities

$$\left|\sum_{|\alpha|=m} A_\alpha(x,\xi_\gamma)\eta_\alpha\right| \leq K \sum_{|\alpha|=m} a_\alpha |\xi_\alpha|^{p_\alpha - 1} |\eta_\alpha| + b_m$$

hold with some constants $K > 0$, $a_\alpha \geq 0$, $p_\alpha > 1$, $b_m \geq 0$, the sequence

p_α being bounded and $b_0 + b_1 + \ldots < \infty$;

b) for any $x \in G$, ξ_γ and ξ_α, $|\alpha| = m$, $|\gamma| \leqq m$ the inequalities

$$\operatorname{Re} \sum_{|\alpha|=m} A_\alpha(x,\xi_\gamma)\overline{\xi}_\alpha \geqq \delta \sum_{|\alpha|=m} a_\alpha |\xi_\alpha|^{p_\alpha} - \delta_m$$

hold with some constants $\delta > 0$, $\delta_m > 0$ where $\delta_0 + \delta_1 + \ldots < \infty$.

Moreover, we assume that the numbers $a_\alpha \geqq 0$, $p_\alpha > 1$ are such that the space $\overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\}$ is non-trivial.

The space of the right-hand sides $W^{-\infty}\{a_\alpha,p'_\alpha\}$, $p'_\alpha = p_\alpha/(p_\alpha - 1)$ is defined as the formal adjoint of the space $\overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\}$. Thus we have

$$W^{-\infty}\{a_\alpha,p'_\alpha\} = \{h(x) : h(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha h_\alpha(x)\} \quad ,$$

where $h_\alpha(x) \in L_{p'_\alpha}(G)$ and

$$\rho'(h) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha(x)\|_{p'_\alpha}^{p'_\alpha} < \infty .$$

The duality of the spaces $W^{-\infty}\{a_\alpha,p'_\alpha\}$ and $\overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\}$ is defined by the relation

$$\langle h,v\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_G h_\alpha(x) D^\alpha v(x)\,dx$$

which easily verified to be correct. Two elements $h_1, h_2 \in W^{-\infty}\{a_\alpha,p'_\alpha\}$ are considered identical if the values $\langle h_1,v\rangle$, $\langle h_2,v\rangle$ coincide for each $v \in \overset{\circ}{W}\{a_\alpha,p_\alpha\}$.

Evidently, we have

$$L(u) : \overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\} \to W^{-\infty}\{a_\alpha,p'_\alpha\}$$

in virtue of the condition a).

DEFINITION 4.1. *A function* $u(x) \in \overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\}$ *is a solution of the Cauchy-Dirichlet problem* (4.1), (4.2) *if* $\langle L(u),v\rangle = \langle h,v\rangle$ *provided* $v(x) \in \overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\}$.

THEOREM 4.1. *Let the conditions a), b) be fulfilled. Then to any right-hand side* $h(x) \in W^{-\infty}\{a_\alpha,p'_\alpha\}$ *there exists a solution* $u(x) \in \overset{\circ}{W}{}^\infty\{a_\alpha,p_\alpha\}$ *of the problem* (4.1), (4.2).

Example. Let us consider the problem

$$L(u) \equiv \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha}|D^{\alpha}u|^{p-2}D^{\alpha}u) = h(x), \quad x \in G,$$

$$D^{\omega}u|_{\Gamma} = 0, \quad |\omega| = 0,1,\dots .$$

If the coefficients are such that the space $\overset{o}{W}{}^{\infty}\{a_{\alpha},p\}$ is non-trivial (e.g. $a_{\alpha} = [(2\alpha)!]^{-p}$, $p > 1$ in the case $\dim G = 1$), then all the assumptions of the theorem are fulfilled.

Counterexample. Let us consider the problem

$$\exp(-\Delta)u(x) = h(x), \quad x \in G ,$$

$$D^{\omega}u|_{\Gamma} = 0, \quad |\omega| = 0,1,\dots .$$

It is easily seen that the corresponding space $\overset{o}{W}{}^{\infty}\{.,2\}$ is empty which means that the problem reduces to the trivial identity $0 = 0$.

5. Behaviour of solutions of nonlinear elliptic equations when the order grows to infinity

In a domain $G \in \mathbb{R}^{\nu}$ with a boundary Γ let us consider a family of Dirichlet problems for nonlinear equations

$$L_{2m}(u_m) \equiv \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha m}(x, D^{\gamma}u_m) = h_m(x) , \tag{5.1}$$

$$D^{\omega}u_m|_{\Gamma} = 0, \quad |\omega| \leqq m - 1 \tag{5.2}$$

where $m = 1,2,\dots$.

We are interested in the behaviour of the problem (5.1), (5.2) when $m \to \infty$. Two cases are possible : the case of a limit equation of infinite order and the case of a limit equation of a finite order; the difference between them is essential.

I. The case of a limit equation of infinite order. Let us assume that the following conditions are fulfilled :

a) The functions $A_{\alpha m}(x,\xi_{\gamma})$ are continuous with respect to $x \in G$ and ξ_{γ} , and for all values of the variables x, ξ_{γ}, η_{α} satisfy the inequalities

$$\left|\sum_{|\alpha|=0}^{m} A_{\alpha m}(x,\xi_\gamma)\eta_\alpha\right| \leq K \sum_{|\alpha|=0}^{m} a_{\alpha m}(|\xi_\alpha|^{p_{\alpha m}-1}+1)|\eta_\alpha|$$

where $K > 0$ is a constant and $a_{\alpha m} \geq 0$, $p_{\alpha m} \geq 1$ are number sequences, the latter being bounded.

b) For any ξ_γ and $x \in G$ the inequality

$$\mathrm{Re} \sum_{|\alpha|=0}^{m} A_{\alpha m}(x,\xi_\gamma)\,\overline{\xi}_\alpha \geq \delta \sum_{|\alpha|=0}^{m} a_{\alpha m}|\xi_\alpha|^{p_{\alpha m}} - K$$

holds with some constants $\delta > 0$, $K > 0$.

c) If $m \to \infty$, then $a_{\alpha m} \to a_\alpha$, $p_{\alpha m} \to p_\alpha$ and $a_\alpha > 0$ for infinitely many values of α. Besides, if $\xi_{\gamma m} \to \xi_\gamma$ in this case, then

$$A_{\alpha m}(x,\xi_{\gamma m}) \to A_\alpha(x,\xi_\gamma)$$

uniformly with respect to $x \in G$ and $A_\alpha(x,\xi_\gamma)$ are continuous functions of their arguments.

d) The space $\overset{\circ}{W}{}^{\infty}\{b_\alpha, q_\alpha\}$ with $b_\alpha = \sup\limits_m a_{\alpha m}$, $q_\alpha = \sup\limits_m p_{\alpha m}$ is non-trivial.

e) The right-hand sides $h_m(x)$ are of the form

$$h_m(x) = \sum_{|\alpha|=0}^{m} a_{\alpha m} D^\alpha h_{\alpha m}(x)$$

where $h_{\alpha m}(x) \in L_{p'_{\alpha m}}(G)$ and

$$\sum_{|\alpha|=0}^{m} a_{\alpha m} ||h_{\alpha m}(x)||^{p'_{\alpha m}}_{p'_{\alpha m}} \leq K$$

holds with a constant $K > 0$ for all m and all values $p_{\alpha m} > 1$ (in the case $p_{\alpha m} = 1$ the inequality $\mathrm{vrai\ max}|h_{\alpha m}| < \delta$ must hold, where $\delta > 0$ is the constant from the condition b)). Moreover, the sequence $h_m(x)$ converges with $m \to \infty$ to $h(x) \in W^{-\infty}\{a_\alpha, p'_\alpha\}$ in the sense that $\langle h_m, v\rangle \to \langle h, v\rangle$ for each function $v(x) \in \overset{\circ}{W}{}^{\infty}\{b_\alpha, q_\alpha\}$.

THEOREM 5.1. *Let the conditions a) - e) be fulfilled. Then the sequence* $u_m(x)$ *of solutions of the problem* (5.1), (5.2) *has a limit point* $u(x)$ *(in the sense of convergence in* $C_0^\infty(G)$ *).*

Moreover, $u(x) \in \overset{o}{W}{}^{\infty}\{a_\alpha, p_\alpha\}$ *and* u *is a solution of the Cauchy-Dirichlet problem*

$$L(u) \equiv \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x) ,$$

$$D^\omega u|_\Gamma = 0, \quad |\omega| = 0,1\dots .$$

Example. Let us consider the problem

$$\sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (a_{\alpha m} |D^\alpha u_m|^{p_{\alpha m}-2} D^\alpha u_m) = h_m(x) ,$$

$$D^\omega u_m|_\Gamma = 0 , \quad |\omega| \leqq m - 1.$$

Let us assume that the sequences $a_{\alpha m} \geqq 0$ and $p_{\alpha m} \geqq 1$ are decreasing with m and $a_{\alpha m} \to a_\alpha$, $p_{\alpha m} \to p_\alpha$. Let these limits be such that the space $\overset{o}{W}{}^{\infty}\{a_\alpha, p_\alpha\}$ is non-trivial. Taking into account the conditions of Theorem 5.1, $b_\alpha = \sup_m a_{\alpha m} = a_\alpha$, $q_\alpha = \sup_m p_{\alpha m} = p_\alpha$ and, consequently, $\overset{o}{W}{}^{\infty}\{b_\alpha, q_\alpha\} = \overset{o}{W}{}^{\infty}\{a_\alpha, p_\alpha\}$. Therefore, in virtue of Theorem 5.1, the solutions $u_m(x)$ converge in $C_0^\infty(G)$ to a solution of the Cauchy-Dirichlet problem

$$\sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} D^\alpha (a_\alpha |D^\alpha u|^{p_\alpha-2} D^\alpha u) = h(x) ,$$

$$D^\omega u|_\Gamma = 0, |\omega| = 0,1,\dots .$$

II. The case of a limit equation of finite order. Again we consider the problem (5.1), (5.2). Let us assume that the above conditions a) - d) are again fulfilled, however, with the following additional assumption: There is an integer $r \geqq 0$ such that $a_\alpha \equiv 0$ provided $|\alpha| \geqq r + 1$. Moreover, let us assume the functions $h_m(x)$ to be in the form

$$h_m(x) = \sum_{|\alpha|=0}^{r} D^\alpha h_{\alpha m}(x)$$

where $h_{\alpha m}(x) \in L_{s_\alpha'}(G)$, $s_\alpha' = \sup_m p_{\alpha m}'$. Further, let $||h_{\alpha m}(x) - h_\alpha(x)||_{s_\alpha'} \to 0$ when $m \to \infty$ with functions $h_\alpha(x) \in L_{s_\alpha'}(G)$.

Finally, let us assume that the operators $L_{2m}(u)$ are monotone for all m , i.e., the inequality

$$\mathrm{Re}\,\langle L_{2m}(u) - L_{2m}(v),\ u - v\rangle \geqq 0$$

holds for any functions $u \in \overset{\circ}{W}{}^{m}_{\vec{p}_m}$, $v \in \overset{\circ}{W}{}^{m}_{\vec{p}_m}$, $\vec{p}_m = \{p_{\alpha m},\ |\alpha| \leqq m\}$.

THEOREM 5.2. *Let the assumptions specified above be fulfilled. Then the family of solutions* $u_m(x)$ *of the problem* (5.1), (5.2) *has a weak limit point* $u(x)$ *with respect to the metric of the space* $\overset{\circ}{W}{}^{r}_{\vec{s}}$ *where* $\vec{s} = \{s_\alpha = \inf_m p_{\alpha m},\ |\alpha| \leqq r\}$. *Further,* $u(x) \in \overset{\circ}{W}{}^{r}_{\vec{p}}$ *and* u *is a solution of the Dirichlet problem*

$$L(u) \equiv \sum_{|\alpha|=0}^{r} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x) ,$$

$$D^\omega u|_\Gamma = 0,\ |\omega| \leqq r - 1$$

with $h(x) = \sum_{|\alpha|=0}^{r} D^\alpha h_\alpha(x)$.

6. Non-homogeneous Dirichlet problem of infinite order

Let a nonlinear elliptic problem

(6.1) $$L(u) \equiv \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x),\ |\gamma| \leqq |\alpha| ,$$

(6.2) $$D^\omega u|_\Gamma = f_\omega(x'),\ x' \in \Gamma ,\ |\omega| = 0,1,\ldots$$

satisfy the assumptions from Section 4. Moreover, let the boundary conditions (6.2) admit an extension in $W^\infty\{a_\alpha, p_\alpha\}$.

THEOREM 6.1. *Under the above specified assumptions, the problem* (6.1), (6.2) *has for any* $h(x) \in W^{-\infty}\{a_\alpha, p'_\alpha\}$ *at least one solution* $u(x) \in W^\infty\{a_\alpha, p_\alpha\}$.

References

[1] MANDELBROJT, S. : Series adhérentes. Regularisation des suites. Applications. Paris 1952.

[2] DUBINSKIĬ, Yu. A. : Sobolev spaces of infinite order and the behaviour of solutions of some boundary-value problems when the order of the equations grows to infinity. Mat. Sb. 98(140), No. 2(10)(1975), 163-184 (Russian).

[3] DUBINSKIĬ, Yu. A. : Non-triviality of Sobolev spaces of infinite order in the case of the whole Euclidean space and of the torus. Mat. Sb. 100(142), No. 3(7)(1975), 436-446 (Russian).

[4] DUBINSKIĬ, Yu. A. : On traces of functions in Sobolev spaces of infinite order and on non-homogeneous boundary value problems. Dokl. Akad. Nauk SSSR 235, No. 6 (1977), 1249-1252 (Russian).

[5] NIKOL'SKIĬ, S. M. : Approximation of functions of several variables and imbedding theorems. Springer-Verlag. Berlin-Heidelberg--New York 1975.