Lars-Erik Persson

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GENERALIZATIONS OF SOME CLASSICAL INEQUALITIES AND THEIR APPLICATIONS

Lars Erik Persson, Luleå

0. INTRODUCTION

The main aim of this talk is to review, complement and generaliz. some inequalities I recently obtained partly together with other authors. The paper is organized in the following way: In section 1 we collect some necessary notations and definitions. We also present some of the classical inequalities we will generalize, unify and complement later on. In section 2 we prove two general "setvalued" inequalities, which in particular generalize some results recently obtained by J. Peetre and the present author [25]. The singlevalued versions of these inequalities are studied in section 3 and several applications are pointed out e.g. a recent result by J. Matkowski [22]. In section 4 we present a generalization of Hölder's inequality to the case with a family of spaces, which recently has been obtained by L. Nikolova and the present author [24]. We also include a generalized completely symmetric form of Hölder's inequality, thereby generalizing some previous results by Aczél-Beckenbach [1] and the present author [29]. In section 5 we prove a sharp generalized form of Minkowski's inequality. For the proof of this inequality we need to prove a certain generalization of Clarcson's inequality, which can be of independent interest. Section 6 is used to prove some relations between generalized versions of some classical inequalities. In particular we find that some of these inequalities, in a sense, are equivalent. In section 7 we present a recent result by T. Koski and the present author [15], where in particular the sharpest possible upper bounds for the exponential entropies $E(\alpha;f)$ are obtained for every $\alpha > 0$, $\alpha \neq 1$. In particular by letting $\alpha \to 1$ we obtain a quite new proof of the "differential entropy inequality", which is one of the corner stones in Information Theory. In section 8 we give some concluding remarks. In particular we present a new results concerning the best constant in a variant of Hardy's inequality. We also include some remarks concerning recent development of the theory and applications of generalized Gini means. In particular we point out some examples of inequalities which may be seen as limiting cases of classical inequalities.

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1. PRELIMINARIES

In this paper we let p and q denote real numbers such that $0 < p < \infty$ and $q = p/(p-1)$. (Ω,Σ,μ) denotes a σ-finite complete measure space and $L^0(\Omega)$ is the space of all complex-valued μ-measurable functions on Ω. Moreover, X denotes a Banach function space on (Ω,Σ,μ), abbreviated BFS, i.e. $X = (X,\|\cdot\|_X)$ is a Banach subspace of $L^0(\Omega)$ satisfying that, for every $x,y \in L^0(\Omega)$, the following lattice property holds:

$$y \in X,\ |x| \le |y|\ \mu\text{-a.e.} \Rightarrow x \in X \text{ and } \|x\|_X \le \|y\|_X .$$

Let A be a Banach space and X a Banach function space on Ω and let $\omega = \omega(s)$, $s \in \Omega$, be a positive weight function on Ω. The spaces $X^p(A,\omega)$ consists of all strongly measurable functions (cross-sections) $x = x(s)$ satisfying

$$\|x\|_{X^p(A,\omega)} = (\|(\|(x(s)\|_A\omega(s))^p\|_X)^{1/p} < \infty,$$

see [24]. For the cases $A = R$ and $\omega = 1$ we use the abbreviated notations $X^p(\omega)$ and $X^p(A)$, respectively. In particular the spaces $X^p(R,1)$ coincides with the usual X^p-spaces. It is easy to prove that the X^p-spaces are Banach function spaces if $p \ge 1$ and at least quasi-Banach function spaces for all $p > 0$ (see e.g. [29]) .

Let Γ denote a linear class of real-valued functions $x = x(t)$ defined on a non-empty set E. If $L: \Gamma \to R$, is an <u>isotone linear functional</u> ("isotone" means that for every $x,y \in L$ such that $x(t) \ge y(t)$ on E it holds that $L(x) \ge L(y)$), then the corresponding <u>generalized Gini mean</u> $G(\alpha,\beta;x)$, $-\infty < \alpha,\beta < \infty$, is defined by

$$G(\alpha,\beta;x) = G_L(\alpha,\beta;x) = \begin{cases} (L(x^\alpha)/L(x^\beta))^{1/(\alpha-\beta)} & ,\ \alpha \ne \beta, \\ \\ \exp\left(\left\{\dfrac{d}{da}(\ln L(x^a))\right\}_{a=\alpha}\right) & ,\ \alpha = \beta, \end{cases}$$

whenever $x^\alpha, x^\beta \in \Gamma$ and $0 < L(x^\alpha), L(x^\beta) < \infty$. Some basic facts concerning these means can be found in [24], [25] and [30]. For the case when $L(x) = \Sigma x_k$ $(x_k \ge 0, k = 1,2,\ldots,n)$ these means coincide with the classical Gini means.

Let D denote an additive Abelian semigroup. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in R^n$. We write $\mathbf{u} \le \mathbf{v}$ if $u_1 \le v_1, u_2 \le v_2, \ldots, u_n \le v_n$. We say that $f : D \to R^n$, is <u>subadditive</u> if

(1.1) $f(x + y) \le f(x) + f(y)$

for all $x,y \in D$. If (1.1) holds in the reversed direction, then we say that f is superadditive. If equality holds in (1.1), then we say that f is affine.

$P(\Omega)$ denotes the power set of Ω, i.e. the set of subsets of Ω.

Concerning classical inequalities and its applications we refer to the books [3], [7], [8], [9], [14] and [21]. We finish this section by listing suitable integral forms of some of these inequalities.

1. *Hölder's inequality*: If $p \geq 1$, then

$$(1.2) \quad \|xy\|_{L^1(\Omega)} \leq \|x\|_{L^p(\Omega)} \|y\|_{L^q(\Omega)} .$$

If $p \leq 1$, $p \neq 0$, $xy > 0$ a.e., then (1.2) holds in the reversed direction.

2. *Minkowski's inequality*: If $p \geq 1$, then

$$(1.3) \quad \|x+y\|_{L^p(\Omega)} \leq \|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)} .$$

If $0 < p \leq 1$ and $x,y \geq 0$ a.e., then (1.3) holds in the reversed direction.

3. *Jensen's inequality*: Let Φ be a non-negative and convex function on $[0,\infty)$. If $\mu(\Omega) = 1$, then, for every $x \in L^1(\Omega)$,

$$(1.4) \quad \Phi\Big(\int_\Omega |x|\, d\mu\Big) \leq \int_\Omega \Phi(|x|)\, d\mu .$$

If Φ is non-negative and concave, then (1.4) holds in the reversed direction.

4. *Clarcson's inequality*: If $1 < p \leq 2$, then

$$(1.5) \quad \left(\left\|\frac{x+y}{2}\right\|_{L^p(\mu)}\right)^q + \left(\left\|\frac{x-y}{2}\right\|_{L^p(\mu)}\right)^q \leq \left(\frac{1}{2}(\|x\|_{L^p(\mu)})^p + \frac{1}{2}(\|x\|_{L^p(\mu)})^p\right)^{q-1} .$$

If $p \geq 2$, then (1.5) holds in the reversed direction.

For the case $p = 2$ Clarcson's inequality reduces to the usual parallellogram law for L^p-spaces.

5. *The* AG-*mean inequality*: If $x = x(t) > 0$ on Ω, $0 < \mu(\Omega) < \infty$, then

$$(1.6) \quad \exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln x\, d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega x\, d\mu .$$

If $\Omega = \{1,2,\dots\ n\}$, $\mu(k) = 1$ and $x(k) = x_k$, $k = 1,2,\dots,n$, then (1.6) reads

$$\left(\prod_1^n x_k\right)^{1/n} \le \frac{1}{n}\sum_1^n x_k$$

which is the most well-known form of the AG-inequality. We note that (1.6) is a special case both of Jensen's inequality and of the fact that the Gini means $G(\alpha,\beta;x)$ are increasing in α (β fixed) and in β (α fixed), see [25] and our Proposition 10.

6. *Beckenbach-Dresher's inequality*: If $0 \le \beta \le 1 \le \alpha$ or if $0 \le \alpha \le 1 \le \beta$, $\alpha \ne \beta$, then

$$(1.7)\quad \left(\frac{\int_\Omega |x+y|^\alpha d\mu}{\int_\Omega |x+y|^\beta d\mu}\right)^{\frac{1}{\alpha-\beta}} \le \left(\frac{\int_\Omega |x|^\alpha d\mu}{\int_\Omega |x|^\beta d\mu}\right)^{\frac{1}{\alpha-\beta}} + \left(\frac{\int_\Omega |y|^\alpha d\mu}{\int_\Omega |y|^\beta d\mu}\right)^{\frac{1}{\alpha-\beta}}.$$

If $\beta \le 0 \le \alpha \le 1$ or if $\alpha \le 0 \le \beta \le 1$, $\alpha \ne \beta$, then (1.7) holds in the reversed direction.

7. *Hardy's inequalities*: If $p > 1$, then

$$(1.8)\quad \int_0^\infty \left(\frac{1}{t}\int_0^t |x(u)|\,du\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |x(t)|^p\,dt.$$

If $0 < p < 1$, then

$$\int_0^\infty \left(\frac{1}{t}\int_t^\infty |x(u)|\,du\right)^p dt \ge \left(\frac{p}{1-p}\right)^p \int_0^\infty |x(t)|^p d\mu.$$

The constants in these inequalities are the best possible.

Limiting cases of some of the inequalities above are pointed out in our section 8. Moreover, the following inequality may be seen as a limiting case of the inequalities we present in our section 7:

8. *The differential-entropy inequality*. Roufly speeking this inequality says that the normal distribution has the largest possible differential-entropy among all distributions with fixed mean and standard deviation. The differential-entropy inequality is described (and generalized) in our section 7. This inequality is maybe not regarded as a classical inequality in the usual textbooks on inequalities but it is very important for applications e.g. in Information Theory (see [4] and [15]).

2. TWO "SETVALUED" GENERALIZATIONS OF SOME CLASSICAL INEQUALITIES

First we state a somewhat generalized form of an inequality of Peetre-Persson[25].

THEOREM 1 *Let* $F: R_+^n \to R_+$ *be non–negative and convex,* $G: D \to P(R_+)$ and $f: D \to R_+^n$. *Then the function*

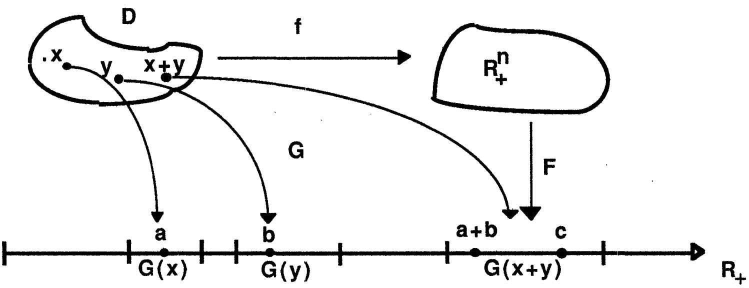
$$f_1(x) = \inf_{a \in G(x)} \left(aF\left(\frac{f(x)}{a} \right) \right)$$

is subadditive if <u>*one*</u> *of the following conditions holds:*

(1) $f(x)$ *is affine and, for all* $a \in G(x)$ *and* $b \in G(y)$, $a+b \in G(x+y)$.

(2) $F(\mathbf{u})$ *is isotone,* $f(x)$ *is subadditive and if* $a \in G(x)$ *and* $b \in G(y)$, *then* $a+b \in G(x+y)$.

(3) $F(\mathbf{0}) = \mathbf{0}$, $F(\mathbf{u})$ *is isotone,* $f(x)$ *is subadditive and, if* $a \in G(x)$ *and* $b \in G(y)$, *then there exists* $c \geq a+b$ *such that* $c \in G(x+y)$.



The proof of Theorem 1 can be carried out by only making obvious modifications of the proof in [25]. Compare also with our proof of the following dual version of Theorem 1:

THEOREM 2 *Let* $F: R_+^n \to R_+$ *be non–negative and concave,* $G: D \to P(R_+)$ *and* $f: D \to P(R_+)$. *Then the function*

$$f_2(x) = \sup_{a \in G(x)} \left(aF\left(\frac{f(x)}{a} \right) \right).$$

is superadditive if <u>*one*</u> *of the conditions* (1)-(3) *in Theorem 1 holds with "subadditive" replaced by "superadditive".*

<u>Proof</u>: First we let the assumptions in the (modified) condition (3) be satisfied. Assume that $x,y \in D$, $a \in G(x)$, $b \in G(y)$ and choose $c \in G(x+y)$ such that $c \geq a+b$. Then

$$\frac{f(x+y)}{c} \geq \frac{f(x)+f(y)}{c} = \frac{a}{c}\frac{f(x)}{a} + \frac{b}{c}\frac{f(y)}{b}$$

and

$$F\left(\frac{f(x+y)}{c}\right) \ge \frac{a}{c}F\left(\frac{f(x)}{a}\right) + \frac{b}{c}F\left(\frac{f(y)}{b}\right) + \frac{c-b-a}{c}F(0) = \frac{a}{c}F\left(\frac{f(x)}{a}\right) + \frac{b}{c}F\left(\frac{f(y)}{b}\right).$$

Thus, for any fixed ε, $0 < \varepsilon < 1/2$, and some $c \in G(x+y)$, we have

$$cF\left(\frac{f(x+y)}{c}\right) \ge (1-\varepsilon)f_2(x) + (1-\varepsilon)f_2(y)\,.$$

By taking supremum once more and letting $\varepsilon \to 0+$ we find that

$$f_2(x+y) \ge f_2(x) + f_2(y).$$

The proofs of the remaining cases are quite similar.

Remark: The proof shows that Theorem 2 holds also if the condition "$F(0) = 0$" is weakened to "$F(0) \ge 0$".

First we present an application for the extreme case when $G(x) = R_+$ for all $x \in D$.

Example Let $D = R_+^n$, $f(x) = x = (x_1,x_2,\ldots,x_n)$, $G(x) = R_+$ for all $x \in D$ and $F(u) = \max(1,\sum_1^n u_k^p)$, $p \ge 1$. Then

$$f_1(x) = \inf_{a\in R_+}\left(a \max\left(1,\sum_1^n\left(\frac{x_k}{a}\right)^p\right)\right) = (\sum_1^n x_k^p)^{1/p}$$

and Theorem 1 implies that

$$(2.1)\quad (\sum_1^n (x_k+y_k)^p)^{1/p} \le (\sum_1^n (x_k)^p)^{1/p} + (\sum_1^n (y_k)^p)^{1/p}.$$

In a similar way Theorem 2 implies that (2.1) holds in the reversed direction if $0 < p < 1$. Moreover, by instead considering $F(u) = 1 + \sum_1^n \Phi(u_k)$, where $\Phi: R_+ \to R_+$, is convex, we obtain

$$f_1(x) = \inf_{a\in R_+} a\left(1 + \sum_1^n \Phi\left(\frac{x_k}{a}\right)\right)$$

and this is the usual Amemiya description of the Orlicz-norm of x. Thus Theorem 1 also contains a generalization of (2.1) to the corresponding Orlicz-space case.

In the next section we present some applications for the other extreme case $G(x) = \{g(x)\}$ (the singleton case).

3. THE SINGLETON CASE WITH APPLICATIONS

We restate Theorems 1 and 2 for the "single-valued" case.

PROPOSITION 3 *Let* $F: R_+^n \to R_+$ be convex, $f: D \to R_+^n$ *and* $g: D \to R_+$. *Then*

$$(3.1)\quad g(x+y)F\left(\frac{f(x+y)}{g(x+y)}\right) \le g(x)F\left(\frac{f(x)}{g(x)}\right) + g(y)F\left(\frac{f(y)}{g(y)}\right)$$

if <u>*one*</u> *of the following conditions holds:*
(a) $f(x)$ *and* $g(x)$ *are affine.*
(b) $F(u)$ *is isotone,* $f(x)$ *is subadditive and* $g(x)$ *is affine.*
(c) $F(0) = 0$, $F(u)$ *is isotone,* $f(x)$ *is subadditive and* $g(x)$ *is superadditive.*
Moreover (3.1) *holds in the reversed direction if* F *is concave and if* (a), (b) *or* (c) *holds with "subadditive" replaced by "superadditive".*

<u>Proof</u>: Apply Theorems 1 and 2 with $G(x) = \{g(x)\}$.

By applying Proposition 3 with $D = L$, $F(u) = u^p$ and using induction and the well-known subadditivity (superadditive) properties of the functionals $f(x) = (B(x^\alpha))^{1/\alpha}$ and $g(x) = C(x^\beta))^{1/\beta}$ (see e.g. [25, Lemma 1.2] and compare also with our Theorem 6) we obtain the following generalization of Beckenbach-Dresher's inequality (see [2] and [13]):

COROLLARY 1 (Peetre-Persson) *Let* $B,C: \Gamma \to R_+$ *be isotone linear functionals and let the functions* $z_i : E \to R_+$, $i = 1,2,\ldots,N$, *satisfy that* $z_i^\alpha, z_i^\beta, (\sum_1^N z_i)^\alpha, (\sum_1^N z_i)^\beta \in \Gamma$.

(a) *If* $\beta \le 1 \le \alpha$, $\beta \ne 0$, $\alpha \ne \beta$ *and* $p \ge 1$, *then*

$$(3.2)\quad \frac{\left(B\left(\left(\sum_1^N z_i\right)^\alpha\right)\right)^{p/\alpha}}{\left(C\left(\left(\sum_1^N z_i\right)^\beta\right)\right)^{(p-1)/\beta}} \le \sum_1^N \frac{\left(B(z_i^\alpha)\right)^{p/\alpha}}{\left(C(z_i^\beta)\right)^{(p-1)/\beta}}\ .$$

(b) *If* $\alpha,\beta \le 1$ $\alpha,\beta \ne 0$ *and* $0 < p \le 1$, *then* (3.2) *holds in the reversed direction.*

See [25]. Another proof of a special case of a) can also be found in [26].

Another obvious consequence of Proposition 3 (a) is the following statement:

COROLLARY 2 (Matkowski). *A function* $F: R_+^n \to R_+$ *is convex if and only if*

$$(3.3)\quad (x+y)F\left(\frac{x+y}{x+y}\right) \le xF\left(\frac{x}{x}\right) + yF\left(\frac{y}{y}\right)$$

holds for all $x,y \in R_+^n$ *and all* $x,y \in R_+$. *The function* F *is concave if and only if* (3.3) *holds in the reversed direction.*

See [22]. Now by using (3.3), induction and a standard density argument for $L^1(\Omega)$-spaces we also have the following corollary:

COROLLARY 3 *If* $F: R_+^n \to R_+$ *is convex and if* $f_1, f_2, \ldots, f_n, g \in L^1(\Omega)$, *then*

$$(3.4)\quad \int_\Omega g\,d\mu\, F\left(\frac{\int_\Omega f_1 d\mu}{\int_\Omega g\,d\mu}, \ldots, \frac{\int_\Omega f_n d\mu}{\int_\Omega g\,d\mu}\right) \le \int_\Omega F\left(\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right) g\,d\mu.$$

This inequality holds in the reversed direction if F *is concave* ($f_i \ge 0$, $g > 0$).

Compare with [22]. We finish this section by pointing out some examples of applications of the results stated in this section.

Example 1 By applying Corollary 1 with $B = C$ and $p = \alpha/(\alpha-\beta)$ and using symmetry we find that

$$G_B(\alpha,\beta;x+y) \le G_B(\alpha,\beta;x) + G_B(\alpha,\beta;y)$$

if $0 \le \beta \le 1 \le \alpha$ or if $0 \le \alpha \le 1 \le \beta$. This inequality holds in the reversed direction if $\beta \le 0 \le \alpha \le 1$ or if $\alpha \le 0 \le \beta \le 1$.

Remark For the case $B(x) = \int_\Omega x\, d\mu$ the statement in Example 1 coincides with the version of *Beckenbach–Dresher's inequality* we presented in the introduction.

It is obviously possible to iterate the procedure in Proposition 3. In particular by using the special case considered in Example 1 we obtain

Example 2 If B and C are isotone linear functionals, if $0 \le \beta \le 1 \le \alpha$, $\gamma \le 0 \le \delta \le 1$ and if $F: R_+ \to R_+$, is convex, isotone and $F(0) = 0$, then

$$G_C(\gamma,\delta;x+y)F\left(\frac{G_B(\alpha,\beta;x+y)}{G_C(\gamma,\delta;x+y)}\right) \le G_C(\gamma,\delta;x)F\left(\frac{G_B(\alpha,\beta;x)}{G_C(\gamma,\delta;x)}\right) + G_C(\gamma,\delta;y)F\left(\frac{G_B(\alpha,\beta;y)}{G_C(\gamma,\delta;y)}\right).$$

In particular if $F(u) = u^p$, $p = (\alpha-\beta)/(\alpha-\beta-\gamma+\delta)$, and $B(x) = C(x) = \int_\Omega x(t)dt$ we obtain the following somewhat curious extention of Beckenbach–Dresher's inequality (1.7): If $0 \le \beta \le 1 \le \alpha$, $\gamma \le 0 \le \delta \le 1$, $\alpha \ne \beta$, $\gamma \ne \delta$, $\alpha-\beta-\gamma+\delta \ge 0$, $x(t), y(t) \ge 0$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu \int_\Omega (x+y)^\delta d\mu}{\int_\Omega (x+y)^\beta d\mu \int_\Omega (x+y)^\gamma d\mu}\right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} \le \left(\frac{\int_\Omega x^\alpha d\mu \int_\Omega x^\delta d\mu}{\int_\Omega x^\beta d\mu \int_\Omega x^\gamma d\mu}\right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} + \left(\frac{\int_\Omega y^\alpha d\mu \int_\Omega y^\delta d\mu}{\int_\Omega y^\beta d\mu \int_\Omega y^\gamma d\mu}\right)^{\frac{1}{\alpha-\beta-\gamma+\delta}}$$

Example 3 Let $\mu(\Omega) = 1$ and $g = 1$. Then, according to Corollary 3, we obtain the following n–dimensional version of *Jensen's inequality* : If $F: R_+^n \to R_+$ is convex and if $\mu(\Omega) = 1$, then

$$F(\int_\Omega f_1 d\mu, \ldots, \int_\Omega f_n d\mu) \le \int_\Omega F(f_1, \ldots, f_n) d\mu.$$

The inequality holds in the reversed direction if F is concave.

Example 4 Let $F(u_1, u_2, \ldots, u_n) = u_1^{1/p_1} u_2^{1/p_2} \ldots u_n^{1/p_n}$, where $\sum_1^n \frac{1}{p_i} = 1$ $(p_i > 1)$. We note that F is concave and, thus, by Corollary 3 used with f_i replaced by $|f_i|^{p_i}$, we obtain that

$$(\int_\Omega |f_1|^{p_1} d\mu)^{1/p_1} (\int_\Omega |f_1|^{p_2} d\mu)^{1/p_2} \ldots (\int_\Omega |f_1|^{p_n} d\mu)^{1/p_n} \ge \int_\Omega |f_1 f_2 \ldots f_n| d\mu.$$

We conclude that also this n-dimensional version of *Hölder's inequality* is a special case of Corollary 3.

Example 5 Let $F(u_1, u_2, \ldots, u_n) = (u_1^{1/p_1} + 1)^{p_1} (u_2^{1/p_2} + 1)^{p_2} \ldots (u_n^{1/p_n} + 1)^{p_n}$, where $p_i > 1$. Since F is concave we can use Corollary 3 to obtain the following generalized version of *Minkowski's inequality* :

$$\int_\Omega \prod_1^n (f_i + g^{1/p_i})^{p_i} \left(\frac{g}{\int_\Omega g d\mu} \right)^{1-n} d\mu \le \prod_1^n ((\int_\Omega f_i^{p_i} d\mu)^{1/p_i} + (\int_\Omega g d\mu)^{1/p_i})^{p_i}.$$

By replacing g by $|f_0|^p$ and putting $n = 1$ and $p = p_1$ in this inequality we obtain the version of Minkowski's inequality we presented in our introduction.

Remark More information concerning the close connection between convexity and inequalities can be found in the forthcoming paper [20].

4. TWO GENERALIZATIONS OF HÖLDER'S INEQUALITY

Let p(t) and W(t) denote positive functions on (0,b), b > 0, and let p be defined by

$$\frac{1}{p} = \frac{1}{I_W} \int_0^b \frac{1}{p(t)} W(t) dt, \quad \text{where } I_W = \int_0^b W(t) dt < \infty.$$

Assume that $\{\omega_t(s)\}$, $t \in [0,b)$, $s \in \Omega$, is a family of weight functions such that $\omega_t(s)$ is measurable on $[0,b) X \Omega$ and define (the generalized geometric mean)

$$\omega = \omega(s) = \exp\left(\frac{1}{I_W} \int_0^b \log \omega_t(s) W(t) dt \right).$$

First we present a generalization of Hölder's inequality to a case where infinite many spaces is involved.

THEOREM 4 (Nikolova-Persson). *Let* X *be a BFS such that* $X'' \equiv X$ and let $Z_t = X^{p(t)}(A,\omega_t)$. *Assume that* $x_t(s) \in Z_t$ *and* $W(t)\log \| x_t(s) \|_{Z_t}$ *and* $W(t)\log \| x_t(s) \|_A$, $s \in \Omega$, *are integrable over* $[0,b)$. *Then*

$$y = y(s) = \exp\left(\frac{1}{I_W} \int_0^b \log \| x_t(s) \|_A W(t)dt \right)$$

belongs to $X^p(\omega)$ *and*

$$\| y \|_{X^p(\omega)} \leq \exp\left(\frac{1}{I_W} \int_0^b \log \| x_t(s) \|_{Z_t} W(t)dt \right).$$

Theorem 1 was discovered by us in connection to our work with interpolation between infinite many spaces (see [23]). A surprisingly simple proof has also been presented in [24].

Remark. We don't know if Theorem 1 holds also without the assumption $X'' \equiv X$. However the proof presented in [24] shows that the theorem holds also without this assumption if we instead assume that $(\alpha_t(s)\omega_t(s))^{p(t)}, t \in [0,b)$, are strongly measurable functions (taking values in X).

COROLLARY. *Let* $z_i \in X^{p_i}(A)$, $0 < p_i < \infty$, $0 < \alpha_i < 1$, $i = 1,2,\ldots,N$, $\sum_1^N \alpha_i = 1$ *and* $\frac{1}{p} = \sum_1^N \frac{\alpha_i}{p_i}$. *Then*

$$\left\| \prod_1^N (\| z_i(s) \|_A)^{\alpha_i} \right\|_{X^p} \leq \prod_1^N (\| z_i(s) \|_{X^{p_i}(A)})^{\alpha_i}.$$

Proof: Let $0 = a_1 < a_2 < \ldots < a_{N+1} = b$ where $\alpha_i = (a_{i+1} - a_i)/b$, $i = 1,2,\ldots,N$.
Apply Theorem 4 and the last remark with $\omega(s) \equiv 1$, $W(t) \equiv 1$, $p(t) = p_i$ and $x_t(s) = z_i(s)$ on $[a_i, a_{i+1})$, $i = 1,2,\ldots,N$, and the proof follows.

Example. *In particular, if* $A = R$, $p_i/\alpha_i = q_i$, $|z_i|^{\alpha_i} = |x_i|$, $i = 1,2,\ldots,N$, *then the estimate in Corollary* 1 *can be written as*

$$\left\| \prod_1^N x_i \right\|_{X^p} \leq \prod_1^N \| x_i \|_{X^{q_i}},$$

where $\frac{1}{p} = \sum_1^N \frac{1}{q_i}$, $0 < q_i < \infty$.

Remark. Concerning this well-known version of Hölder's inequality for finite many X^p-spaces see e.g. [18] and the references given there.

Remark. It is also possible to obtain other classical inequalities as special cases of Theorem 4 e.g. the generalizations of Minkowski's and Beckenbach-Dresher's inequalities presented in [29] (see [24]).

In this section we also include the following generalized and completely symmetric version of Hölder's inequality:

THEOREM 5 *Let* p,q,r *be real numbers* $\neq 0$ *such that* $1/p + 1/q = 1/r$ *and let* x *and* y *denote positive functions. Let* $B_p(x) = (B(x^p))^{1/p}$, *where* B *denotes an isotone linear functional defined on* x *and* y *and suitable potenses of these functions. Then*

(a $\quad B_r(xy) \leq B_p(x)B_q(y)$

if $p > 0, q > 0, r > 0$ *or if* $p < 0, q > 0, r < 0$ *or if* $p > 0, q < 0, r < 0$.

(b) $\quad B_r(xy) \geq B_p(x)B_q(y)$

if $p > 0, q < 0, r > 0$ *or if* $p < 0, q > 0, r > 0$ *or if* $p < 0, q < 0, r < 0$.

Theorem 5 can be proved by using similar arguments as those in the proof of the special case considered in [29] (compare also with [1]). For the sake of completeness we include a proof (which essentially only depends on the convexity of the exponential function).

Proof: First we consider the case $p > 0, q > 0, r > 0$. Let $B_p(x) = B_q(y) = 1$. Since $f(t) = \exp(t)$ is convex we have

$$|x(t)y(t)|^r = \exp(r(\ln|x(t)| + \ln|y(t)|)) = \exp\left(\frac{r}{p}\ln|x(t)|^p + \frac{r}{q}\ln|y(t)|^q\right) =$$

$$= \frac{r}{p}|x(t)|^p + \frac{r}{q}|y(t)|^q.$$

Thus, according to the fact that B is a linear and isotone functional, we find that

$$(B_r(xy))^r \leq \frac{r}{p}B_p^p(x) + \frac{r}{q}B_q^q(y) = \frac{r}{p} + \frac{r}{q} = 1, \text{ i.e. } B_r(xy) \leq 1.$$

For the general case we use this estimate with x and y replaced by $x/B_p(x)$ and $y/B_q(y)$, respectively. Next we let $p > 0, q < 0, r > 0$. By using the estimate we just have proved we find that

$$B_p(x) = B_p\left(xy\frac{1}{y}\right) \leq B_r(xy)B_{-q}\left(\frac{1}{y}\right) = B_r(xy)(B_q(y))^{-1}, \text{ i.e. (b) holds.}$$

By symmetry we find that (b) holds also for the case $p < 0, q > 0, r > 0$. For the cases $p < 0, q > 0, r < 0$, and $p > 0, q < 0, r < 0$, we use the obtained results with r,p,q replaced by $-r,-p,-q$, respectively, and obtain

$$B_r(xy) = (B_{-r}((xy)^{-1})^{-1} \leq (B_{-p}(x^{-1}))^{-1}\,(B_{-q}(y^{-1}))^{-1} = B_p(x)B_q(y), \text{ i.e. (b) holds.}$$

The proof of the remaining case $p < 0, q < 0, r < 0$, is quite similar.

Remark By using the relation $B_r(x) = (B_{-r}(x^{-1}))^{-1}$ and induction we see that Theorem 5 implies that if

$$\prod_1^n z_i = 1, \sum_1^n \frac{1}{p_i} = 0, p_i \neq 0, i = 1,2,\ldots ,n,$$

and if exactly one of p_i is negative (respectively positive), then

$$\prod_1^n B_{p_i}(z_i) \geq 1 \text{ (respectively } \prod_1^n B_{p_i}(z_i) \leq 1).$$

A SHARP GENERALIZED VERSION OF MINKOWSKI'S INEQUALITY

Let B denote an isotone and linear functional defined on the positive functions x, y, x^p and y^p. We introduce the following measure of the "deviation" between x and y:

$$d(x,y) = B_p\left(\frac{x}{B_p(x)} - \frac{y}{B_p(y)}\right), B_p(x) = (B(|x|^p))^{1/p}, \ 0 < B_p(x), B_p(y) < \infty.$$

We note that $0 \leq d(x,y) \leq 2$, $a(x,y) = 0$ iff $y = cx$, $c > 0$, and $d(x,y) = 2$ iff $y = cx$, $c < 0$. In particular this means that the coefficients b_i in the formulation of our next theorem satisfies $-1 \leq b_i \leq 1$ and thus that the theorem may be regarded as a sharp generalized form of Minkowski's inequality.

THEOREM 6 *Let* $0 < p < \infty$ *and let* y_i *be functions such that* $0 < B_p(y_i) < \infty$. If $y = \sum_1^n y_i$,

$a_i = d(y_i, y)$, $b_i = 2\left(1-\left(\frac{a_i}{2}\right)^q\right)^{1/q} - 1$ *if* $1 < p \leq 2$ *and* $b_i = 2\left(1-\left(\frac{a_i}{2}\right)^p\right)^{1/p} - 1$ *if* $p \geq 2$,

$i = 1,2,\ldots ,n$, *then*

(5.1) $\quad B_p(\sum_1^n y_i) \leq \sum_1^n b_i B_p(y_i)$.

Remark We can define $\theta_i = \arccos b_i$ and compare (5.1) with the well-known formula

$$\| \sum_1^n x_i \|_{R^3} = \sum_1^n b_i \|x_i\|_{R^3}, \ b_i = \cos \theta_i,$$

where x_i are vectors in R^3 and θ_i are the angles between x_i and $\sum_1^n x_i$, $i = 1,2,\ldots ,n$.

Example Let $0 < \varepsilon < 1$ and $y = -(1-\varepsilon)x$. Then an elementary calculation shows that $a_1 = d(x,x+y) = 0$, i.e. $b_1 = 1$, and $a_2 = d(y,x+y) = 2$, i.e. $b_2 = -1$. Thus, by Theorem 6, $B_p(x+y) \leq B_p(x) - B_p(y)$ if $y = -(1-\varepsilon)x$ (for the L^p-case we have even equality).

For the proof of Theorem 6 we need the following crucial lemma:

LEMMA 1 *Let* Y *be a uniformly convex space,* $y_i \in Y$, $y = \sum_1^n y_i$, $a_i = d(y_i, y)$, $i = 1,2, \dots ,n$, *where* $d(x,y) = \| \frac{x}{\|x\|_Y} - \frac{y}{\|y\|_Y} \|_Y$. *Then*

$$\| \sum_1^n y_i \|_Y \leq \sum_1^n (1-2\delta(a_i)) \| y_i \|_Y ,$$

where $\delta(\varepsilon)$ *denotes the modulus of convexity in the definition of uniformly convex spaces.*

A proof of Lemma 1 can be found in Clarcson's original paper [10].

We also need the following generalization of Clarcson's inequality, which can be of independent interest

LEMMA 2 *If* $1 < p \leq 2$ *and if* B *is an isotone and linear functional, then*

$$B_p^q\left(\frac{x+y}{2}\right) + B_p^q\left(\frac{x-y}{2}\right) \leq \left(\frac{1}{2}B_p^p(x) + \frac{1}{2}B_p^p(y)\right)^{q-1} .$$

If $p \geq 2$, *then this inequality holds in the reversed direction.*

The proof of this lemma can be carried out by only making suitable modifications of the proof for the special case where $B_p(x) = \|x\|_X{}^p$ (see [29]) so we omit the details.

More information concerning Clarcson's inequality can also be found in [11], [12] and [19] (and the references given in these papers).

<u>Proof of Theorem 6</u>. Let B^p, $p > 1$, denote the normed space equipped with the norm $\|x\|_{B^p} = B_p(x)$. Let $0 < \varepsilon < 1/2$, $\|x\|_{B^p} = \|y\|_{B^p} = 1$ and $\|x-y\|_{B^p} \geq \varepsilon > 0$. If $1 < p \leq 2$, then, by Lemma 2, we find that

$$\left(\| \frac{x+y}{2} \|_{B^p} \right)^q + \left(\frac{\varepsilon}{2}\right)^q \leq 1, \text{ i.e. } \| \frac{x+y}{2} \|_{B^p} \leq 1-\delta,\ \delta = 1 - \left(1-\left(\frac{\varepsilon}{2}\right)^q\right)^{1/q} .$$

In the same way we find that if $p \geq 2$, then

$$\| \frac{x+y}{2} \|_{B^p} \leq 1-\delta,\ \delta = 1 - \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} .$$

Thus we have proved that B^p is a uniformly convex space where the corresponding modulus of convexity $\delta = \delta(\varepsilon)$ can be chosen as above. Finally, we apply Lemma 1 and the proof follows.

We finish this section by pointing out the following special case of Theorem 6:

Example If $1 < p \le 2$, $y_i \in L^s(\mu)$, $p \le s \le q$, $d_i = d(y_i, \sum_1^n y_i)$ and $b_i = 2\left(1-\left(\frac{a_i}{2}\right)^q\right)^{1/q} -1$, $i = 1,2,\ldots,n$, then

$$\|\sum_1^n y_i\|_{L^s(\mu)} \le \sum_1^n b_i \|y_i\|_{L^s(\mu)}.$$

For the cases $s = p$ and $s = q$ this statement can also be found in [10, p.405]. See also [29].

6. RELATIONS BETWEEN GENERALIZED VERSIONS OF SOME CLASSICAL INEQUALITIES

Let $B,C : \Gamma \to R_+$ denote arbitrary isotone linear functionals. In this section we present some relations between the following generalized forms of classical inequalities, which hold simultanously for all isotonic linear functionals.

Hölder's inequality (H): If $p > 1$, then $B(xy) \le B_p(x)B_q(y)$.
Lyaponov's inequality (L) : If $0 < \gamma < \beta < \alpha$, then

$$B(x^\beta) \le (B(x^\gamma))^{(\alpha-\beta/(\alpha-\gamma)}(B(x^\alpha)^{(\beta-\gamma)/(\alpha-\gamma)}.$$

Generalized AG-inequality (P): $G_B(\alpha,\beta;x)$ is nondecreasing in α (β fixed) and in β (α fixed).
See [25] and our Proposition 10. In particular the special case $G_B(0,0;x) \le G_B(1,0;x)$ is a generalization of the usual AG-inequality.
Minkowski's inequality (M): If $p \ge 1$, then $B_p(x+y) \le B_p(x) + B_p(y)$.
Beckenbach-Dresher's inequality (BD): If $p \ge 1$ and $\beta \le 1 \le \alpha$, $\beta \ne 0$, then

$$\frac{(B((x+y)^\alpha))^{p/\alpha}}{(C((x+y)^\beta))^{(p-1)/\beta}} \le \frac{(B(x^\alpha))^{p/\alpha}}{(C(x^\beta))^{(p-1)/\beta}} + \frac{(B(y^\alpha))^{p/\alpha}}{(C(y^\beta))^{(p-1)/\beta}}.$$

Remark By choosing $p = \alpha/(\alpha-\beta)$, $B = C$ and using the symmetry and continuity properties of $G_B(\alpha,\beta;x)$ we find that (BD) implies that the inequality

(p) $\quad G_B(\alpha,\beta;x+y) \le G_B(\alpha,\beta;x) + G_B(\alpha,\beta;y)$

holds if $0 < \beta \le 1 \le \alpha$ or if $0 < \alpha \le 1 \le \beta$.

THEOREM 7 *The following holds:*

$$\begin{array}{c} (M) \\ \Uparrow \\ (P) \Leftrightarrow (H) \Leftrightarrow (L) \\ \Downarrow \\ (BD) \end{array}$$

Remark Theorem 7 may be seen as a generalization of some results in [21, p. 457].

Remark According to Theorem 7 and our proof of Theorem 5 we see that all the inequalities (H), (L), (P), (M), (BD), and (p) easily follows from the convexity of the function exp(t).

Proof: (P) $\Rightarrow$ (H). Assume that (P) holds. Let $0 < \theta < 1$ and consider the functional $B_0(x) = B(x\omega)$, where ω denotes a positive weight function. Then

$$\frac{1}{B(\omega)} B\left(\left(\frac{z}{\omega}\right)^{\theta} \omega\right) = \frac{B_0\left(\left(\frac{z}{\omega}\right)^{\theta}\right)}{B_0(1)} = \left(G_{B_0}\left(\theta,0;\frac{z}{\omega}\right)\right)^{\theta} \leq \left(G_{B_0}\left(1,0;\frac{z}{\omega}\right)\right)^{\theta} = \left(\frac{B_0\left(\frac{z}{\omega}\right)}{B_0(1)}\right)^{\theta} =$$

$$(B(z))^{\theta}(B(\omega))^{-\theta} \text{ i.e. } B(z^{\theta}\omega^{1-\theta}) \leq (B(z))^{\theta}(B(\omega))^{1-\theta}.$$

We put $x = z^{\theta}w^{1-\theta}$, $y = w^{1-\theta}$, $\theta = 1/p$ and find that (H) holds.

(H) $\Rightarrow$ (L). This implication follows by using (H) with $x = z^{\gamma/p}$ and $y = z^{\alpha/q}$, where $\frac{1}{p} = \frac{\alpha-\beta}{\alpha-\gamma}$ and $\frac{1}{q} = \frac{\beta-\gamma}{\alpha-\gamma}$.

(L) $\Rightarrow$ (P). Let (L) hold. We define $F(a) = B(x^a)$, $a \in R$, and note that

$$F(\beta) \leq (F(\gamma))^{\theta}(F(\alpha))^{1-\theta}, \quad \theta = \frac{\alpha-\beta}{\alpha-\gamma}, \quad \gamma < \beta < \alpha \text{ or } \alpha < \beta < \gamma.$$

This means that

$$G_B(\alpha,\beta;x) = \left(\frac{F(\alpha)}{F(\beta)}\right) \geq \left(\frac{F(\alpha)}{F(\gamma)}\right)^{\frac{1}{\alpha-\gamma}} = G_B(\alpha,\gamma;x) \text{ if } \gamma < \beta < \alpha$$

and

$$G_B(\alpha,\beta;x) = \left(\frac{F(\alpha)}{F(\beta)}\right) \leq \left(\frac{F(\alpha)}{F(\gamma)}\right)^{\frac{1}{\alpha-\gamma}} = G_B(\alpha,\gamma;x) \text{ if } \alpha < \beta < \gamma$$

Hence by using the symmetry and continuity properties of $G_B(\alpha,\beta;x)$ (see [25]) we find that (P) holds.

(H) $\Rightarrow$ (M). Write $(x+y)^p = (x+y)^{p-1}x + (x+y)^{p-1}y$, apply B and use (H).

(H) $\Rightarrow$ (BD) Let α_1, α_2, β_1, β_2 denote positive numbers. According to (H) (applied with $B(\alpha) = \alpha_1 + \alpha_2$) we have

$$(6.1) \quad \frac{(\alpha_1+\alpha_2)^p}{(\beta_1+\beta_2)^{p-1}} \leq \frac{\alpha_1^p}{\beta_1^{p-1}} + \frac{\alpha_2^p}{\beta_2^{p-1}}.$$

Since (H) $\Rightarrow$ (M) we also have

(6.2) $(B((x+y)^{\alpha}))^{1/\alpha} \le (B(x^{\alpha}))^{1/\alpha} + (B(y^{\alpha}))^{1/\alpha}$

and

(6.3) $(C((x+y)^{\beta}))^{1/\beta} \ge (C(x^{\beta}))^{1/\beta} + (C(y^{\beta}))^{1/\beta}$.

By using (6.1)-(6.3) we obtain that

$$\frac{(B((x+y)^{\alpha}))^{p/\alpha}}{(C((x+y)^{\beta}))^{(p-1)/\beta}} \le \frac{((B(x^{\alpha}))^{1/\alpha} + (B(y^{\alpha}))^{1/\alpha})^{p}}{((C(x^{\beta}))^{1/\beta} + (C(y^{\beta}))^{1/\beta})^{p-1}} \le \frac{(B(x^{\alpha}))^{p/\alpha}}{(C(x^{\beta}))^{(p-1)/\beta}} + \frac{(B(y^{\alpha}))^{p/\alpha}}{(C(y^{\beta}))^{(p-1)/\beta}}.$$

The proof is complete

7. A GENERALIZATION OF THE DIFFERENTIALENTROPY INEQUALITY

Let X be a stochastic variable with the density function $f_X(x) = f(x)$, $-\infty < x < \infty$. Then the differentialentropy H(X) is defined by

$$H(X) = -\int_{-\infty}^{\infty} f(x)\ln f(x)dx.$$

Example If $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{x^2}{2\sigma^2}\right), -\infty < x < \infty$ (the normal distribution), then

$$H(X) = \ln(\sigma\sqrt{2\pi e}).$$

According to the **differential–entropy inequality** it holds that

$$H(X) \le \ln(\sigma\sqrt{2\pi e})$$

for every stochastic variable with the fixed variance σ^2.

We introduce the following **scale** of entropies:

$$H(\alpha;X) = \frac{1}{1-\alpha}\ln\left(\int_{-\infty}^{\infty} (f(x))^{\alpha}dx\right), \alpha > 0, \alpha \ne 1,$$

and note that the continuity property of generalized Gini means (see [25]) implies that

$$\lim_{\alpha \to 1} H(\alpha;X) = H(X).$$

We also define the **exponential entropies** $E(\alpha;X)$ by $E(\alpha;X) = \exp H(\alpha;X)$. We remark that these entropies are of special interest for some important problems in Information Theory e.g. in the theory of quantization of data (see [4], [15] and [17]).

Let $\Gamma(x)$ and $B(p,r)$ denote the usual Gamma and Beta functions, respectively (see e.g. [31]). For convenience of writing we also introduce the function $T(a)$ defined by

$$T(a) = \int_{-1}^{1} (1 + x^2)^a \, dx \, , \; a \in R.$$

The following precize estimates for the exponential entropies can in particular be seen as a generalization of the differential-entropy inequality:

THEOREM 8 (Koski-Persson). *Consider any continuous probability distribution on the real line with the density* $f(x)$ *with finite standard deviation* σ_0.

a) *If* $\alpha > 1$, *then*

$$E(\alpha; f) \le \left(\frac{2\alpha}{3\alpha - 1}\right)^{\frac{1}{1-\alpha}} \sqrt{\frac{3\alpha - 1}{\alpha - 1}} \, B\left(\frac{\alpha}{\alpha - 1}, \frac{1}{2}\right) \sigma_0 .$$

Equality is obtained if and only if f(x) coincides (almost everywhere) with

$$f^*(x) = \begin{cases} \dfrac{c}{a}\left(1 - \dfrac{x^2}{a^2}\right)^{\frac{1}{\alpha - 1}} , & |x| < a, \\ 0 , & |x| \ge a, \end{cases}$$

where $c = \dfrac{1}{B\left(\frac{\alpha}{\alpha - 1}, \frac{1}{2}\right)}$ *and* $a = \sqrt{\dfrac{3\alpha - 1}{\alpha - 1}} \sigma_0 .$

b) *If* $1 > \alpha > 1/3$, *then*

$$E(\alpha; f) \le \left(\frac{2\alpha}{3\alpha - 1}\right)^{\frac{1}{1-\alpha}} \sqrt{\frac{3\alpha - 1}{1 - \alpha}} \, B\left(\frac{1 + \alpha}{2(1 - \alpha)}, \frac{1}{2}\right) \sigma_0 .$$

Equality is obtained if and only if f(x) coincides (almost everywhere) with

$$f^*(x) = \frac{c}{a}\left(1+\frac{x^2}{a^2 r}\right)^{-\frac{r+1}{2}}, \quad -\infty < x < \infty,$$

where $r = \dfrac{1+\alpha}{1-\alpha}$, $c = \dfrac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)}$ *and* $a = \sqrt{\dfrac{r-2}{r}}\,\sigma_0$.

c) *If* $0 \le a \le 1/3$ *and if* f *satisfies the "tail condition"*

$$f(x) \le \alpha^{\frac{1}{\alpha-1}} \frac{c}{a}\left(1+\frac{x^2}{a^2}\right)^{\frac{1}{\alpha-1}} \quad \textit{for } |x| \ge a,$$

where

$$c = \frac{1}{T\left(\frac{1}{\alpha-1}\right)} \quad \textit{and} \quad a = \sqrt{\frac{T\left(\frac{1}{\alpha-1}\right)}{T\left(\frac{\alpha}{\alpha-1}\right) - T\left(\frac{1}{\alpha-1}\right)}}\,\sigma_0,$$

then

$$E(\alpha; f) \le \left(T\left(\frac{1}{\alpha-1}\right)\right)^{\frac{\alpha}{\alpha-1}} \left(T\left(\frac{\alpha}{\alpha-1}\right)\right)^{\frac{1}{1-\alpha}} \sqrt{\frac{T\left(\frac{1}{\alpha-1}\right)}{T\left(\frac{\alpha}{\alpha-1}\right) - T\left(\frac{1}{\alpha-1}\right)}}\,\sigma_0.$$

Equality is obtained if and only if f(x) coincides (almost everywhere) with

$$f^*(x) = \begin{cases} \dfrac{c}{a}\left(1+\dfrac{x^2}{a^2}\right) & , |x| < a, \\ 0 & , |x| \ge a. \end{cases}$$

A detailed proof of Theorem 7 together with some applications can be found in [15]

Remark. By making some straightforward calculations we find that the constants on the right hand sides of a) and b) both converge to $(2\pi\varepsilon)^{1/2}\,\sigma_0$ if $\alpha \to 1+$ and $a \to 1-$, respectively. Mreover, some other elementary calculations show that the corresponding distributions converge to the distribution function of the normal distribution (see [15]).

Remark. The optimal distributions discussed in this section is mostly of the types classified in the usual Pearson's system of frequency curves. More information about generalized entropies, its applications and relations to generalized Gini means has recently been obtained in [15], [17] and [25].

8. ON THE BEST CONSTANT IN A VARIANT OF HARDY'S INEQUALITY AND CONCLUDING REMARKS

We consider a nonincreasing and nonnegative function $x = x(t)$ on $(0,\infty)$. First we recall the following variant of Hardy's inequality:

If $0 < p,r,s < \infty$ *and* $0 < b < 1/p$, *then there exists a finite constant* $K = K(p,r,s,b)$, *such that*

$$(8.1)\quad \int_0^\infty t^{-sb+s/p}\left(\int_t^\infty (x(u)u^b)^r \frac{du}{u}\right)^{s/r}\frac{dt}{t} \le K\int_0^\infty (x(t)t^{1/p})^s\frac{dt}{t}.$$

A proof of (a weighted version of) this estimate can be found in [28] (see also [27]). The best possible constant is not known in general. However, the following result was recently proved by J. Bergh [5]:
If $0 < p < 1$, *then*

$$(8.2)\quad \int_0^\infty\left(\frac{1}{t}\int_t^\infty x(u)du\right)^p dt \le \frac{\pi p}{\sin\pi p}\int_0^\infty (x(t))^p dt.$$

The constant $\pi p/\sin\pi p$ *cannot be improved.*

Here we present the following generalization of this statement:

PROPOSITION 9 *If* $r = 1$, *then the best possible constant in* (8.1) *is equal to*

$\frac{1}{b^s}\frac{s}{p}B\left(\frac{s}{p} - bs, 1+bs\right)$. *Equality occur if* $x(t) \equiv \chi_{[0,c]}$ *for any* $c > 0$.

Proof The proof of Proposition 9 can be carried out by only making obvious modifications of the proof by J. Bergh so we omit the details.

Remark For the case $b = 1$ also V.I. Burenkov [6] has independently (and with a quite different proof) obtained the result in Proposition 9 (and, thus, Bergh's result). For the case $s = p$, $b = 1$, Proposition 9 coincides with Bergh's result.

Some new information concerning Generalized Gini means have been obtained in [24], [25] and [30]. Here we mention only the following result which we partly already have used previously in this paper:

PROPOSITION 10 (Peetre-Persson) *Let* $-\infty < \alpha,\beta < \infty$ *and consider the generalized Gini means* $G(\alpha,\beta;x) = G_A(\alpha,\beta;x)$. *Then*

(a) $\quad G(\alpha,\beta;x) = \exp\dfrac{1}{\alpha-\beta}\displaystyle\int_\beta^\alpha \ln G(a,a;x)da.$

(b) $G(\alpha,\beta;x)$ *is non-decreasing in* α (β *fixed*) *and in* β (α *fixed*).

(c) $G(\alpha,\beta;x)$ *is continuous in* α *and* β.

A proof of the representation formula in (a) can be found in [25]. One simple proof of (b) (and, thus, of a generalized AG-inequality) only consists of using the representation formula and a generalized form of Schwarz´ inequality (see [25]).

Here we only remark that it is in many cases easy to obtain appropriate limiting cases of known inequalities by using these continuity and monotonity properties of Generalized Gini means. For example by replacing $|x(t)|$ by $|x(t)|^p$ in Hardy´s inequality (1.8) by considering $G_A\left(\frac{1}{p},0;x\right)$ with $A(x) = \int_0^t x(u)du$ we obtain in this way the usual *Carleman´s inequality*

$$\int_0^\infty \exp\left(\frac{1}{t}\int_0^t \ln|x(u)|du\right)dt \le e\int_0^\infty |x(t)|dt$$

as a limiting case (of (1.8)) as $p \to \infty$. In a similar way we find that if x(t) has its support on [0,1], then the inequality

$$\int_0^1 \ln\frac{x(t)}{1-t}dt \le \int_0^1 \ln x'(t)\, dt$$

may be regarded as a limiting case of (8.2) as $p \to 0$ (for a simple direct proof of this inequality see also [5]). Moreover, the limiting case of (1.7) for $\alpha = \beta = 1$ (and positive x(t) and y(t)) reads:

$$\exp\left(\frac{\int_\Omega (x+y)\ln(x+y)d\mu}{\int_\Omega (x+y)d\mu}\right) \le \exp\left(\frac{\int_\Omega x\ln x d\mu}{\int_\Omega x d\mu}\right) + \exp\left(\frac{\int_\Omega y\ln y d\mu}{\int_\Omega y d\mu}\right),$$

and the corresponding inequality for the other extremal case $\alpha = \beta = 0$ reads:

$$\exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln(x+y)d\mu\right) \ge \exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln x d\mu\right) + \exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln y d\mu\right).$$

As special cases of these inequalities we obtain the following inequalities for positive sequences:

$$\left(\prod_1^n (x_k+y_k)^{x_k+y_k}\right)^{1/\sum_1^n (x_k+y_k)} \le \left(\prod_1^n x_k^{x_k}\right)^{1/\sum_1^n x_k} + \left(\prod_1^n y_k^{y_k}\right)^{1/\sum_1^n y_k}$$

and

$$(\prod_1^n(x_k+y_k))^{1/n} \geq (\prod_1^n x_k)^{1/n} + (\prod_1^n y_k)^{1/n}$$

respectively. For another proof of the last inequality see also [3,p.26].

It is well-known that several inequalities can be proved by using interpolation. Some new results in this connection concerning classical inequalities and its applications can be found in [16] and [19]. Finally we remark that we in our sections 2 and 3 have given examples of applications of our general inequalities in Theorems 1 and 2 only for the extremal cases. Sigrid Sjöstrand, Lund (personal communication) has recently pointed out to me some new applications also in "intermediate" cases. However, this new interesting possibility seems not to be fully investigated yet.

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