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Some Properties of Step-Punctions Connected with Extensions of Measures

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A step-function is any real-valued function whose range is (at most) countable. We discuss some measurability properties of step-functions formulated in terms of extensions of measures. The case of invariant (quasiinvariant) measures is considered especially. We show that this case essentially differs from the case of ordinary measures.

Let E be a nonempty set and let f be a function acting from E into the real line $\mathbf{R}$.

We recall that f is a step-function if the range of f is (at most) countable.

Clearly, every step-function $f : E \rightarrow \mathbf{R}$ produces a countable partition

$$\{X_i : i \in I\} = \{f^{-1}(t) : t \in ran(f)\}$$

of E. Conversely, let $\{X_i : i \in I\}$ be an arbitrary countable partition of E. We shall say that a step-function $f : E \rightarrow \mathbf{R}$ is associated with this partition if the following relations are satisfied:

(a) the restriction of f to any set X_i is constant;

(b) the restriction of f to any selector of $\{X_i : i \in I\}$ is an injection.

It is well known that step-functions with additional properties play an important role in many topics of mathematical analysis, especially, in those ones which are connected with various kinds of approximations. For instance, if E is equipped

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with a σ-finite measure μ, then μ-measurable step-functions are necessary for introducing the class of μ-integrable real-valued functions. Also, step-functions are essentially used in some questions concerning the sup-measurability of functions of two variables, and so on.

Below, the symbol μ^* (respectively, μ_*) denotes the outer measure (respectively, the inner measure) associated with a σ-finite measure μ given on E.

Let $f : E \to \mathbf{R}$ be a μ-measurable step-function. It can easily be verified that, for any subset T of $\mathbf{R}$, the pre-image $f^{-1}(T)$ is a μ-measurable subset of E. It is natural to conjecture that this property is tightly connected with the notion of a step-function. Indeed, in the sequel it will be shown that a similar measurability property enables to characterize step-functions in terms of extensions of measures.

We shall say that a function $f : E \to \mathbf{R}$ is strongly measurable with respect to μ if, for any $T \subset \mathbf{R}$, the pre-image $f^{-1}(T)$ is a μ-measurable subset of E.

One can readily check that, for any step-function $f : E \to \mathbf{R}$, the following three relations are equivalent:

(c) f is measurable with respect to μ;

(d) f is strongly measurable with respect to μ;

(e) for each $t \in \mathbf{R}$, the pre-image $f^{-1}(t)$ is measurable with respect to μ

For our further purposes, we need two auxiliary propositions.

Lemma 1. *Let μ be a σ-finite measure on E and let $\{X_i : i \in I\}$ be an arbitrary disjoint family of subsets of E. Then there exists a measure μ' on E extending μ and satisfying the relation*

$$\{X_i : i \in I\} \subset dom(\mu').$$

In particular, if $\{X_i : i \in I\}$ is a countable partition of E, then every step-function associated with $\{X_i : i \in I\}$ becomes measurable with respect to μ'.

For a simple proof of this lemma, see [1].

We recall that a subset T of $\mathbf{R}$ is universal measure zero if, for any nonzero σ-finite diffused Borel measure ν on $\mathbf{R}$, we have $\nu^*(T) = 0$, where ν^* denotes the outer measure associated with ν. The next classical result is well known in descriptive set theory.

Lemma 2. *There are uncountable universal measure zero subsets of* $\mathbf{R}$.

Note that constructions of uncountable universal measure zero subsets of $\mathbf{R}$ were presented by various authors and different ideas were used in those constructions (the existence of a canonical decomposition of a proper analytic set into its Borel components, Marczewski's characteristic function, Ulam's transfinite matrix, Fubini type argument etc.). For more details, see, e.g., [2], [7], [10], [11], [15].

Theorem 1. *Let E be an uncountable set and let $f : E \to \mathbf{R}$ be a function. The following two assertions are equivalent:*

(1) f is a step-function;

(2) for any σ-finite measure μ on E, there exists a measure μ' on E extending μ and such that f is strongly measurable with respect to μ'.

Proof. (1) $\Rightarrow$ (2). Assume that (1) is valid and consider an arbitrary σ-finite measure μ on E. Let $\{Y_i : i \in I\}$ denote the countable partition of E produced by f. According to Lemma 1, there exists an extension μ' of μ such that

$$\{Y_i : i \in I\} \subset dom(\mu').$$

Obviously, f is strongly measurable with respect to μ', so (2) is true.

(2) $\Rightarrow$ (1). Assume that (2) is valid and let us show that f is a step-function. Suppose otherwise, i.e. $card(f) \geq \omega_1$. Clearly, we can find a set $X \subset E$ such that $f|X$ is an injection and $card(ran(f|X)) = \omega_1$. Consider a complete diffused probability measure μ on E which is concentrated on X (the existence of μ is obvious). By virtue of (2), there exists an extension μ' of μ such that f becomes strongly measurable with respect to μ'. Now, for any set $Z \subset ran(f|X)$, let us put $\nu(Z) = \mu'(f^{-1}(Z))$. So we get a diffused probability measure ν which is defined on the family of all subsets of $ran(f|X)$. From the existence of ν we easily conclude that there is no universal measure zero subset of **R** whose cardinality equals $card(ran(f|X)) = \omega_1$. But this contradicts Lemma 2.

Under some additional set-theoretical assumptions, Theorem 1 can be strengthened. For instance, let us consider the following set-theoretical assertion:

(*) Any uncountable subset of **R** contains an uncountable universal measure zero set.

It can easily be seen that (*) is implied by the conjunction of Martin's Axiom with the negation of the Continuum Hypothesis, so (*) is consistent with **ZFC** theory. On the other hand, the existence of a Sierpiński subset of **R** readily implies that (*) is false. So, (*) is independent of **ZFC**.

The next statement is valid.

Theorem 2. *Suppose (*). Let E be an uncountable set and let $f : E \rightarrow \mathbf{R}$ be a function. The following two assertions are equivalent:*

(1) f is a step-function;

(2) for any σ-finite measure μ on E, there exists a measure μ' on E extending μ and such that f is measurable with respect to μ'.

Proof. The argument is very similar to the proof of Theorem 1. The implication (1) $\Rightarrow$ (2) does not need an additional set-theoretical assumption and can be established in the same manner as above. Assume now that (2) is satisfied and show that f is a step-function. Again, suppose otherwise, i.e. $card(ran(f)) \geq \omega_1$. According to (*), there exists an uncountable universal measure zero set $Y \subset ran(f)$. Clearly, we can find a set $X \subset E$ such that $ran(f|X) = Y$ and the restriction $f|X$ is an injection. Consider in E an arbitrary complete diffused probability measure μ which is concentrated on the set X. By virtue of (2), there exists an extension μ' of μ such that f becomes μ'-measurable. Now, for every Borel subset Z of Y, let us put $\nu(Z) = \mu'(f^{-1}(Z))$. A straightforward verification shows that ν turns out to be a Borel diffused probability measure on Y, so we obtain a contradiction with the fact that Y is a universal measure zero subset of $\mathbf{R}$. This ends the proof of the theorem.

In particular, Theorem 2 implies that it is impossible to define, within **ZFC** theory, a non-step-function $f : E \to \mathbf{R}$ having the measurability property (2).

Now, let us consider step-functions for those nonzero σ-finite measures on E, which are invariant (or, more generally, quasiinvariant) with respect to an uncountable group of transformations of E, which acts freely in E. Here the situation is absolutely different. To see this, take an arbitrary uncountable set E with $cf(card(E)) = \omega$. Let G be a group of transformations of E acting freely in E and such that $card(E) = card(G)$. Fix a countable partition $\{X_i : i \in I\}$ of E satisfying the relations $card(X_i) < card(E)$ for all $i \in I$. Let $f : E \to \mathbf{R}$ be a step-function associated with this partition. It is easy to verify that f cannot be measurable with respect to a nonzero σ-finite G-quasiinvariant measure on E.

To give more deep examples of this kind, let us introduce two definitions.

Let E be a set and let G be a group of transformations of E.

We say that a set $X \subset E$ is G-absolutely nonmeasurable if, for any nonzero σ-finite G-quasiinvariant measure μ on E, we have $X \notin dom(\mu)$.

We say that a set $Y \subset E$ is G-absolutely negligible if, for every σ-finite G-invariant (G-quasiinvariant) measure μ on E, there exists a G-invariant (G-quasiinvariant) extension μ' of μ such that $\mu'(Y) = 0$.

Some properties of G-absolutely nonmeasurable and G-absolutely negligible subsets of E are discussed in [5] and [6].

In a particular case, where E is an uncountable commutative group (identified with the group G of all its translations), the following statement is valid.

Lemma 3. *If $(G, +)$ is an uncountable commutative group (or, more generally, an uncountable solvable group), then there exists a G-absolutely nonmeasurable subset of G and there exists a countable partition of G into G-absolutely negligible sets.*

For the proof of Lemma 3, see [5] and [6]. From this lemma we get two examples.

Example 1. Let X be a G-absolutely nonmeasurable subset of an uncountable commutative group $(G, +)$ and let f_X be its characteristic function (which trivially is a step-function). Then f_X is nonmeasurable with respect to any nonzero σ-finite G-quasiinvariant measure on G.

In other words, Example 1 states that there are two-valued functions absolutely nonmeasurable with respect to the class of all nonzero σ-finite G-quasiinvariant measures on G.

Example 2. For any uncountable commutative group $(G, +)$, consider its countable partition $\{Y_i : i \in I\}$ into G-absolutely negligible sets. Let $f : G \to \mathbf{R}$ denote any step-function associated with this partition. Then:

(1) f is nonmeasurable with respect to every nonzero σ-finite G-quasiinvariant measure on G;

(2) for each $t \in \mathbf{R}$, the set $f^{-1}(t)$ is G-absolutely negligible.

Indeed, (2) is obvious. To see (1), let us suppose that f is measurable with respect to some nonzero σ-finite G-quasiinvariant measure μ. Then all sets Y_i, being the pre-images of certain singletons, must be μ-measurable. Since all of them are also G-absolutely negligible, we must have $\mu(Y_i) = 0$ whence it follows that

$$\mu(G) = \mu(\cup\{Y_i : i \in I\}) = \sum\{\mu(Y_i) : i \in I\} = 0,$$

which yields a contradiction.

In other words, Example 2 states that there exist step-functions f on an uncountable commutative group $(G, +)$, which are absolutely nonmeasurable with respect to the class of all nonzero σ-finite G-quasiinvariant measures on G, but each of the pre-images $f^{-1}(t)$ $(t \in \mathbf{R})$ is good for extending any σ-finite G-invariant (G-quasiinvariant) measure μ on G.

However, there are certain types of step-functions which are good for obtaining invariant (quasiinvariant) extensions of invariant (quasiinvariant) measures. To describe such functions, return to the general situation when a set E is given with some group G of its transformations. For our purpose, the notion of an almost G-invariant subset of E turns out to be helpful. This notion was first considered by E. Marczewski [13] (see also [9]). It plays an important role in various topics of the theory of invariant and quasiinvariant measures (see, e.g., [3], [4], [5], [12], [13]). There are two definitions of almost invariant sets, which are rather similar to each other.

A set $Z \subset E$ is called to be almost G-invariant in E (in the set-theoretical sense) if $card(g(Z) \triangle Z) < card(E)$ for each $g \in G$.

If μ is a measure on E, then a set $Z \subset E$ is called to be almost G-invariant with respect to μ if $\mu^*(g(Z) \triangle Z) = 0$ for each $g \in G$.

Notice that if a set $Z \subset E$ is almost G-invariant with respect to μ, then any measurable hull of Z is also almost G-invariant with respect to μ.

The next lemma is probably well known (cf. [3], [4], [12]) but, for the sake of completeness, we present its short proof here.

Lemma 4. *Let E be an uncountable set, G be a group of transformations of E with $card(G) \leq card(E)$, and let I be a nonempty countable set. Then there exists a partition $\{X_i : i \in I\}$ of E consisting of almost G-invariant subsets of E such that $card(X_i) = card(E)$ for all $i \in I$.*

Proof. We may assume, without loss of generality, that $card(G) = card(E)$ and G acts transitively in E. Let α denote the least ordinal for which $card(\alpha) = card(E)$

and let x be a fixed point of E. An increasing (by inclusion) transfinite sequence $\{G_\xi : \xi < \alpha\}$ of subgroups of G can easily be constructed satisfying the following conditions:

(1) $\cup\{G_\xi : \xi < \alpha\} = G$;

(2) $card(G_\xi) \leq card(\xi) + \omega$ for any $\xi < \alpha$;

(3) $G_\xi(x) \setminus \cup\{G_\zeta(x) : \zeta < \xi\} \neq \emptyset$ for any $\xi < \alpha$.

Now, let $\{\Xi_i : i \in I\}$ be a partition of α such that $card(\Xi_i) = card(\alpha)$ for each $i \in I$. Putting

$$X_i = \cup\{(G_\xi(x) \setminus \cup\{G_\zeta(x) : \zeta < \xi\}) : \xi \in \Xi_i\} \qquad (i \in I),$$

we come to the required partition $\{X_i : i \in I\}$ of E.

Lemma 5. *Let μ be a σ-finite G-quasiinvariant measure on E and let $f : E \to \mathbf{R}$ be a step-function such that, for any $t \in \mathbf{R}$, the set $f^{-1}(t)$ is almost G-invariant with respect to μ. Then, for any $g \in G$, the functions f and $f \circ g$ are equivalent with respect to μ, i.e. the equality*

$$\mu^*(\{x \in E : f(x) \neq (f \circ g)(x)\}) = 0$$

holds true.

Proof. Denote by $\{X_i : i \in I\}$ the countable partition of E associated with f. It is clear that

$$\{x \in E : f(x) \neq (f \circ g)(x)\} \subset \cup\{X_i \cap g^{-1}(X_j) : i \in I, j \in I, i \neq j\}.$$

Since the relations

$$\mu^*(g^{-1}(X_j) \triangle X_j) = 0, \qquad X_i \cap X_j = \emptyset \qquad (i \neq j)$$

are satisfied, we must have

$$\mu^*(X_i \cap g^{-1}(X_j)) = 0.$$

From this, taking into account the countability of I, we immediately obtain the required result.

Theorem 3. *Let E be a set with $cf(card(E)) > \omega$, let G be a group of transformations of E which acts freely in E and whose cardinality is equal to $card(E)$, and let $\{X_i : i \in I\}$ be a countable partition of E into almost G-invariant sets. Denote by $f : E \to \mathbf{R}$ any step-function associated with this partition. Then, for every σ-finite G-invariant (G-quasiinvariant) measure μ on E, there exists a G-invariant (G-quasiinvariant) measure μ' on E such that:*

(1) μ' extends μ;

(2) f is measurable with respect to μ'.

Proof. Since I is countable, we may suppose that either $I = \{1,2,\dots, n\}$ or $I = \omega$. For any $i \in I$, *denote by* t_i the value of f at some point of X_i.

Let μ be an arbitrary σ-finite G-invariant (G-quasiinvariant) measure on E. Since $cf(card(E)) > \omega$, we may assume, without loss of generality, that any set $Z \subset E$ with $card(Z) < card(E)$ belongs to the domain of μ and, consequently, $\mu(Z) = 0$. Therefore, all sets $X_i (i \in I)$ become almost G-invariant with respect to μ.

For each index $i \in I$, denote by Y_i a μ-measurable hull of X_i and define

$$Z_i = Y_i \setminus \cup \{Y_j : j < i\}.$$

Notice that all sets Z_i are pairwise disjoint, μ-measurable and almost G-invariant with respect to μ. Moreover, we have $E = \cup (Z_i : i \in I\}$. Let $f' : E \to \mathbf{R}$ be a step-function whose value on every nonempty set Z_i is equal to t_i. As established in paper [1],

$$\mu_*(\{x \in E : f(x) \neq f'(x)\}) = 0.$$

But here we need a much stronger property of the set

$$V = \{x \in E : f(x) \neq f'(x)\}.$$

Namely, we must show that, for any countable family $\{g_k : k < \omega) \subset G$, the equality

$$\mu_*(\cup \{g_k(V) : k < \omega\}) = 0$$

holds true. Indeed, the inclusion

$$\cup \{g_k(V) : k < \omega\} \subset V \cup \{\{x \in E : (f \circ g_k^{-1})(x) \neq f(x)\} : k < \omega\}$$
$$\cup \{\{x \in E : (f' \circ g_k^{-1})(x) \neq f'(x)\} : k < \omega\}$$

is easily verified. Taking into account the relation $\mu_*(V) = 0$ and applying Lemma 5, we claim that $\mu_*(\cup \{g_k(V) : k < \omega\}) = 0$.

Thus, the set V generates a G-invariant σ-ideal of subsets of E, all whose members are of inner μ-measure zero. This circumstance enables us to extend the given measure μ to a G-invariant (G-quasiinvariant) measure μ' on E such that $\mu'(V) = 0$ (cf. [13]). Since the function f' is μ-measurable, it is also μ'-measurable. In view of the equality $V = \{x \in E : f(x) \neq f'(x)\}$, we conclude that f is μ'-measurable, too.

Remark 1. Actually, the preceding argument shows that if we have a σ-finite G-invariant (G-quasiinvariant) measure μ on E and a step-function $f : E \to \mathbf{R}$ such that all pre-images $f^{-1}(t) (t \in \mathbf{R})$ are almost G-invariant with respect to μ, then there exists a G-invariant (G-quasiinvariant) extension μ' of μ for which f becomes μ'-measurable.

In connection with Theorems 1 and 2, the following question naturally arises: does there exist a real-valued function $f_1 : E \to \mathbf{R}$ with $card(ran(f_1)) > \omega$ such that every σ-finite G-invariant (G-quasiinvariant) measure μ on E admits a G-invariant (G-quasiinvariant) extension for which f_1 becomes measurable?

To answer positively to this question, we need the next auxiliary proposition.

Lemma 6. *Let E be a set and let G be an uncountable group of transformations of E acting freely in E. Then there exists an uncountable G-absolutely negligible subset X of E.*

Proof. If $card(G) > \omega_1$, then any set $X \subset E$ with $card(X) = \omega_1$ is G-absolutely negligible in E. Suppose now that $card(G) = \omega_1$ and fix a point $x \in E$. Let $\{G_\xi : \xi < \omega_1\}$ be an increasing (by inclusion) ω_1-sequence of subgroups of G satisfying the following relations:

(a) $\cup(G_\xi : \xi < \omega_1\} = G$;
(b) $card(G_\xi) \leq \omega$ for each $\xi < \omega_1$;
(c) $G_\xi(x) \setminus \cup \{G_\zeta(x) : \zeta < \xi\} \neq \emptyset$ for each $\xi < \omega_1$.

Let X be a selector of the family of nonempty sets

$$\{(G_\xi(x) \setminus \cup \{G_\zeta(x) : \zeta < \xi\}) : \xi < \omega_1\}.$$

Then it is not difficult to verify that X is a G-absolutely negligible subset of E (cf. [5], [6]). Since X is also uncountable, we get the required result.

The next statement readily follows from Lemma 6.

Theorem 4. *Under the assumptions of Lemma 6, there exists a function* $f_1 : E \to \mathbf{R}$ *such that:*

(1) $card(ran(f_1)) = \omega_1$*;*

(2) for any σ-finite G-invariant (G-quasiinvariant) measure μ on E, there exists a G-invariant (G-quasiinvariant) extension of μ for which f_1 *becomes equivalent to zero and, consequently, becomes measurable.*

Proof. Using Lemma 6, we can find a G-absolutely negligible subset X of E with $card(X) = \omega_1$. Let $f_1 : E \to \mathbf{R}$ be a function defined as follows: $f_1|X$ is injective and $f_1|(E \setminus X)$ is equal to zero. A straightforward verification shows that f_1 satisfies relations (1) and (2) of Theorem 4.

Comparing Theorem 2 with Theorem 4, we see that the case of ordinary measures essentially differs from the case of invariant (quasiinvariant) measures. Moreover, in view of Theorem 3, the following natural question arises: how to characterize those step-functions f on E which have the measurability property in the sense that, for any σ-finite G-invariant (G-quasiinvariant) measure μ on E, there exists a G-invariant (G-quasiinvariant) extension μ' of μ such that f becomes μ'-measurable.

Example 3. Let f be a step-function of Theorem 3 such that $ran(f) \cap \{0\} = \emptyset$ and let f_1 be a function of Theorem 4. We put $f_2 = f + f_1$. Then, for any nonzero σ-finite G-invariant (G-quasiinvariant) measure μ on E, there exists a G-invariant (G-quasiinvariant) extension μ' of μ such that f_2 is μ'-measurable and, at the same time, f_2 never becomes equivalent to zero (with respect to μ'). In addition to this, if $card(E) = \omega_1$ and the Continuum Hypothesis holds, then the set $ran(f_2) \subset \mathbf{R}$ can be as bad as possible.

Remark 2. If we deal with finitely additive G-invariant normalized measures on E, then it is reasonable to call a step-function any function $f: E \to \mathbf{R}$ whose range is finite. In this case, measurability properties of f essentially depend on the algebraic structure of G. For instance, if G is amenable and μ is an arbitrary finitely additive G-invariant normalized measure μ on E, then, according to von Neumann's extension theorem, every step-function on E becomes measurable with respect to an appropriate universal finitely additive G-invariant extension μ' of μ. On the other hand, if $E = G$ and G admits paradoxical decompositions, then it is obvious that there exist step-functions on E which are absolutely nonmeasurable with respect to the class of all finitely additive left G-invariant normalized measures on E (cf. Example 1). For more details about paradoxical decompositions, see e.g. [8] and [14].

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