Hurewicz Scheme

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We present a simple proof of Hurewicz theorem saying that every coanalytic non-G_δ-set C in a Polish space contains a countable set $L \subseteq C$ without isolated points such that $(\overline{L} \setminus L) \cap C = \emptyset$.

Hurewicz theorem mentioned in the abstract has many important consequences, e.g., every analytic space with property E^* is σ-compact [2] or the Kechris-Louveau-Woodin Dichotomy Theorem [3]. The original proof by W. Hurewicz [1] based on the notion of a "Häufungsystem" is elementary, however rather complicated. A. Kechris [3] presents a proof based on game theory.

Main goal of this note is a simple elementary proof of a generalization of Hurewicz theorem. Actually we follow the original Hurewicz proof. We shall use common set theoretical terminology and notations, say those of [4]. In the next we assume that (X, ϱ) is a Polish space with a countable base $\mathscr{B} = \{V_n : n \in \omega\}$ of open sets. We use a little modified notion of a "Häufungsystem".

Let ${}^n\omega$ be the set of finite sequences $v = (v(0), \dots, v(n-1))$ of length n from ω. If $v \in {}^n\omega$ and $m \leq n$, we let $v|m = (v(0), \dots, v(m-1))$. Let u be a finite sequence from ω of length at least n. We shall write $v \leq u$ if $v = u|n$.

A mapping $\varphi : {}^{<\omega}\omega \to X$ is called a **Hurewicz scheme** on X if

(1) $(\forall v \in {}^{<\omega}\omega)(\forall m, n \in \omega)(m \neq n \to \varphi(v^\frown m) \neq \varphi(v^\frown n))$,

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(2) $(\forall v \in {}^{<\omega}\omega)\, \varphi(v) = \lim_{n\to\infty} \varphi(v^\frown n)$,
(3) $(\forall v \in {}^{<\omega}\omega)\, \lim_{k\to\infty} \operatorname{diam}\{\varphi(u) : u \geq v^\frown k\} = 0$,
(4) $(\forall f \in {}^{\omega}\omega)\, \lim_{k\to\infty} \operatorname{diam}\{\varphi(u) : u \geq f|k\} = 0$.
The following result describes a basic property of a Hurewicz scheme.

Lemma 1. *If φ is a Hurewicz scheme on X and $x \in \overline{\operatorname{rng}(\varphi)} \setminus \operatorname{rng}(\varphi)$, then there exists a branch $f \in {}^{\omega}\omega$ such that $x = \lim_{k\to\infty} \varphi(f|k)$.*

Proof. Assume that $x \in \overline{\operatorname{rng}(\varphi)} \setminus \operatorname{rng}(\varphi)$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of $\operatorname{rng}(\varphi)$ and $\{u_n\}_{n=0}^{\infty}$ a sequence of elements of ${}^{<\omega}\omega$ such that $x_n \to x$ and $x_n = \varphi(u_n)$, $n \in \omega$. Denote

$$T = \{v \in {}^{<\omega}\omega : (\exists n \in \omega)\, v \leq u_n\}.$$

We show that the tree T has finite branching degree. Assume not, i.e. there exists a node $v \in T$ and an increasing sequence $\{m_k\}_{k=0}^{\infty}$ such that $v^\frown m_k \in T$. Then for every k there exists n_k such that $v^\frown m_k \leq u_{n_k}$. Since

$$\varrho(\varphi(v), \varphi(u_{n_k})) \leq \varrho(\varphi(v), \varphi(v^\frown m_k)) + \operatorname{diam}\{\varphi(u) : u \geq v^\frown m_k\},$$

by (2) and (3) we obtain $x = \lim_{k\to\infty} \varphi(u_{n_k}) = \varphi(v) \in \operatorname{rng}(\varphi)$ – a contradiction.
By König's lemma there is an infinite branch $f \in {}^{\omega}\omega$ for which $\{f|k : k \in \omega\} \subseteq T$. Let n_k be such that $f|k \leq u_{n_k}$. By (4) $\lim_{k\to\infty} \varrho(\varphi(f|k), \varphi(u_{n_k})) = 0$ and therefore $x = \lim_{k\to\infty} \varphi(f|k)$.

q.e.d.

Let A, B be sets such that $A \subseteq B$. A set C separates A, B if $A \subseteq C \subseteq B$.

Lemma 2. *Let $A, B \subseteq X$ and let $U \subseteq X$ be an open set such that $A \cap U \subseteq B$. If $A \cap U, B$ cannot be separated by an F_σ-set, then there exist infinitely many points $p \in U \setminus B$ such that for every neighborhood V of p, the sets $A \cap V, B$ cannot be separated by an F_σ-set either.*

Proof. Assume there is no such point $p \in U \setminus B$. Let

$$S = \{n \in \omega) : (V_n \subseteq U) \wedge (A \cap V_n, B \text{ can be separated by an } \mathrm{F}_\sigma\text{-set})\}.$$

For each $n \in S$ let us choose an F_σ-set F_n which separates $A \cap V_n, B$ and let us denote $W = \bigcup_{n\in S} V_n$ and $F = \bigcup_{n\in S} F_n$. Since $U \setminus B \subseteq W \subseteq U$ and $A \cap W \subseteq F \subseteq$ $\subseteq B$, the F_σ-set $(F \cap U) \cup (U \setminus W)$ separates $A \cap U, B$, what is a contradiction. If U had only finitely many points with the desired property, eliminating them from U you obtain a contradiction.

q.e.d.

Theorem 3. *Let A be an analytic subset of X. For every set B with $A \subseteq B \subseteq X$ the following are equivalent:*
i) *A, B cannot be separated by an F_σ-set,*
ii) *there is a countable $L \subseteq X \setminus B$ without isolated points such that $\overline{L} \setminus L \subseteq A$.*

Proof. Assume that i) holds true. Since A is analytic (see [4]), there exists a closed Suslin scheme ${}^{<\omega}\omega, \psi$, with vanishing diameter such that

$$\bigcup_{f\in{}^{\omega}\omega}\bigcap_{n\in\omega}\psi(f|n) = A.$$

We can assume that $u < v \to \psi(u) \supseteq \psi(v)$ for any $u, v \in {}^{<\omega}\omega$. For every $v \in {}^{<\omega}\omega$ denote

$$A_v = \bigcup\{\bigcap_{n\in\omega}\psi(f|n) : f \in {}^{\omega}\omega \wedge v \subseteq f\}.$$

We construct functions $\varphi : {}^{<\omega}\omega : \to X$ and $F : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that

a) φ is a Hurewicz scheme on $X\setminus B$,

b) F preserves ordering on ${}^{<\omega}\omega$,

c) $\varphi(v) \in \psi(F(v))\setminus A_{F(v)}$ for any $v \in {}^{<\omega}\omega$,

d) there is no F_σ-set separating $A_F(v) \cap U, B$ for any neighborhood U of $\varphi(v)$.

Apply Lemma 2 for $U = X$ and fix $p \in X\setminus B$ from the conclusion. Therefore $p \in \overline{A}\setminus A \subseteq \psi(\emptyset)\setminus A$. We set $\varphi(\emptyset) = p$ and $F(\emptyset) = \emptyset$. Let $s \in {}^{k}\omega$ and $\varphi(s)$, $F(s)$ be already defined and satisfy c), d). Fix $n \in \omega$. Since $A_{F(s)} = \bigcup_m A_{F(s)^\frown m}$ and

$$U = B_\varrho\left(\varphi(s), 2^{-\sum_{i\in dom(s)}(s(i)+1)-n}\right)$$

is a neighbourhood of $\varphi(s)$, there exists an $m \in \omega$ such that $A_{F(s)^\frown m} \cap U, B$ cannot be separated by an F_σ-set. Let $p \in U\setminus B$ be that of Lemma 2. We set $\varphi(s^\frown n) = p$ and $F(s^\frown n) = F(s)^\frown m$. We can assume that $\varphi(s^\frown n)$, $n \in \omega$, are mutually distinct. If $u \geq s^\frown n$ then

$$\varrho(\varphi(s), \varphi(u)) \leq 2^{-\sum_{i\in dom(s)} s(i)-n}.$$

Moreover, by d) we have

$$\varphi(s^\frown n) \in \overline{A_{F(s^\frown n)}} \subseteq \psi(F(s^\frown n)).$$

Thus, φ is a Hurewicz scheme on $X\setminus B$ such that

$$\lim_{n\to\infty}\varphi(f|n) \in \bigcap_{n\in\omega}\psi(F(f|n)) \subseteq A,$$

for any branch $f \in {}^{\omega}\omega$ and thus the set $L = \mathrm{rng}(\varphi)$ satisfies ii).

Assume that i) does not hold true while ii) does. Then there exists a G_δ-set G separating $X\setminus B$, $X\setminus A$. Since G is a Polish subspace L is not closed in G and therefore $(\overline{L}\setminus L) \cap G \neq \emptyset$, which is a contradiction.

q.e.d.

Corollary 4 (W. Hurewicz). *If C is a coanalytic non-G_δ-set in a Polish space then there exists a countable set $L \subseteq C$ without isolated points such that $(\overline{L}\setminus L) \cap C = \emptyset$.*

Proof. Take $A = B = X\backslash C$.

q.e.d.

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