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Lecture Notes On Ideal Dichotomy

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Lectures prepared for the Hejnice Winter School in the Czech Republic, February 2009[1] in which dichotomies of ω_1-generated ideals and P-ideals were explained with some applications and consistency proofs. Our aim here is to develop some of the main ideas rather than to give a complete treatment of the subject, and an Appendix section discusses some background material.

1. Introduction

Todorcevic introduced combinatorial statements that have the form of an ideal dichotomy. Let I be an ideal of countable sets (which means that $I \subset [S]^{\leq\omega}$ for some set S is closed under subsets and finite unions, and, for convenience, we assume that $[S]^{<\omega} \subseteq I$). We say that $X \subseteq S$ is "out of I" (or orthogonal to I) iff $I \cap [X]^{\omega} = \emptyset$. We say that X is "inside I" iff $[X]^{\leq\omega} \subseteq I$. In plain words: out of I means that no infinite subset of X is in I, and inside I means that all countable subsets of X are in I. Again, X is out of I if the restriction of I on X is the ideal of finite sets, and X is inside I if the restriction of I on X is the ideal of *all* countable subsets.

The symmetric form of the dichotomy for a family $\mathscr{F}$ of ideals of countable sets over an uncountable set S is the statement that for every $I \in \mathscr{F}$:

(1) There is an uncountable $X \subseteq S$ which is inside I, or

(2) there is an uncountable $X \subseteq S$ which is out of I.

Asymmetric forms are stronger, and they come in two types:

form 1 : If $I \in \mathscr{F}$ then either there is a decomposition $S = \bigcup_{i\in\omega} S_i$ where each S_i is inside I, or there is an uncountable subset of S that is out of I.

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[1] Some parts of these lectures were presented in the set-theory seminar, Jerusalem, January–February 2007.

form 2 : If $I \in \mathscr{F}$ then either there is a decomposition $S = \bigcup_{i\in\omega} S_i$ where each S_i is out of I, or there is an uncountable subset of S that is inside I.

These dichotomies were introduced by Todorcevic for two families: the ω_1-generated ideals, and the P-ideals. Following Todorcevic [2] , Abraham and Todorcevic [1], and Todorcevic [3], we shall describe here some applications of these dichotomies as well as their consistency which follows from the Proper Forcing Axiom PFA.

We shall first deal with ω_1-generated ideals.

2. Dichotomy for ω_1-generated ideals

An ideal is ω_1-generated if there is a family $\{A_\alpha \mid \alpha < \omega_1\}$ such that I is the collection of all subsets of finite unions of A_α sets.

Two asymmetric forms of ideal dichotomy were introduced by Todorcevic and proved to follow from the Proper Forcing Axiom PFA.

form 1 : If I is any ω_1 generated ideal of countable subsets of S then either there is a decomposition $S = \bigcup_{i\in\omega} S_i$ where each S_i is inside I, or there is an uncountable subset of S that is out of I.

form 2 : If I is any ω_1 generated ideal of countable subsets of S then either there is a decomposition $S = \bigcup_{i\in\omega} S_i$ where each S_i is out of I, or there is an uncountable subset of S that is inside I.

We shall first prove that the PFA implies the first form of the dichotomy, and then we shall prove that the second form is a consequence of the conjunction of the first form with Martin Axiom and the negation of the continuum hypothesis (and hence it is also a consequence of the PFA).

Let I be an ω_1-generated ideal of countable subsets of some set S which we may assume to be ω_1. Assume that I is generated by $\{A_\alpha \mid \alpha < \omega_1\}$, and we may further assume that $A_\alpha \subseteq \alpha$ for every $\alpha \in \omega_1$.

Assume that there is no decomposition of S into countably many sets that are inside I, and we shall show that some proper poset forces an uncountable set outside of I.

Definition of the poset. $p \in P$ iff $p = (x_p, d_p, N^p)$ is such that the following hold.

(1) $N^p = \{N^p_0, \dots, N^p_{k-1}\}$ is a finite set of countable elementary substructures of $H(\aleph_2)$ enumerated such that $N^p_i \in N^p_{i+1}$ for every $i < k-1$. $I \in N^p_0$. ($H(\aleph_2)$ is the structure consisting of all sets of cardinality hereditarily less than $\aleph_2$. See the Appendix for basic facts concerning these structures.)

(2) $x_p \in [\omega_1]^{<\omega}$ is "separated" by N^p. This means that for every two points in x_p, say $\alpha < \beta$, there is a model N^p_i such that $\alpha < N^p_i \cap \omega_1 < \beta$. For notational purposes, it is convenient to assume that x_p contains $k+1$ points. So that if $x_p = \alpha_0 < \cdots < \alpha_k$, then we have $\alpha_0 < N^p_0 \cap \omega_1 < \alpha_1 < N^p_1 \cap \omega_1 \cdots N^p_{k-1} \cap$ $\cap\, \omega_1 < \alpha_k$.

(3) For every α in x_p and structure N_i^p not containing α (we say α is "above N_i") α does not belong to any set in N_i^p that is inside I:

$$\alpha \notin \bigcup\{X \mid X \in N_i^p \text{ is inside } I\}.$$

(4) $d_p \in [\omega_1]^{<\omega}$.

Define $q \leq p$ (q extends p; is more informative) iff

(1) $x_p \subseteq x_q$, $d_p \subseteq d_q$, and $N^p \subseteq N^q$.

(2) For every $\alpha \in d_p$, $x_p \cap A_\alpha = x_q \cap A_\alpha$.

The role of x_p is to generically develop an uncountable set $X = \bigcup\{x_p \mid p \in G\}$ outside of I (where G is the generic filter). For this, we want to make sure that $X \cap A_\alpha$ is finite for every $\alpha < \omega_1$, and the role of d_p is to fix $x_p \cap A_\alpha$ for $\alpha \in d_p$. (This is the reason for requiring in the extension relation that $x_p \cap A_\alpha$ remains fixed. The separation by the models ensures, as we shall see, that P is a proper poset.

Claim 2.1 *For every $p \in P$ and $\gamma \in \omega_1$ there is an extension q of p such that $x_q \setminus \gamma \neq \emptyset$.*

Proof. Suppose $N^p = \{N_0^p, \ldots, N_{k-1}^p\}$. As any condition, $p \in H(\aleph_2)$. So we may find a countable $N_k^p \prec H(\aleph_2)$ with $\gamma, p \in N_k^p$. If the set $X = \omega_1 \setminus \bigcup\{X \mid X \in N_k^p$ is inside $I\}$ is countable, then ω_1 is a countable union of sets that are inside I. Since this is not the case, we can find an ordinal $\alpha \in \omega_1 \setminus X$ and add N_k^p and α to x_p. More formally, we define $(x_p \cup \{\alpha\}, d_p, \{N_0,^p, \ldots, N_{k-1}^p, N_k^p\})$ as the required extension. □

Clearly, it is always possible to add an ordinal to d_α. That is, if p is any condition then $(x_p, d_p \cup \{\alpha\}, N^p)$ is an extension.

So we can define ω_1 dense sets in P ensuring that the filter G that intersects them indeed produces a set $X = \bigcup\{x_p \mid p \in G\}$ that is uncountable and is out of I.

The main point is to prove properness of P, so that the proper forcing axiom can be applied (for details on the PFA see the appendix). Take any regular cardinal $\kappa > 2^{|P|}$, consider the structure $H(\kappa)$ of all sets of cardinality hereditarily less than κ, and let $M \prec H(\kappa)$ be countable. Suppose $p_0 \in M \cap P$ is an arbitrary condition. Define $p = p_0 + (M \cap H(\aleph_2))$ as the condition $(x_{p_0}, d_{p_0}, N^{p_0} \cup \{M \cap H(\aleph_2)\})$. Since $p_0 \in M \cap H(\aleph_2)$, $N_0^{p_0}, \ldots, M \cap H(\aleph_2)$ is an $\in$-sequence and hence p is a condition that extends p_0. We shall prove that p is an M-generic condition over P. For this we must prove that if p' is any extension of p and $D \in M$ is dense in P, then p' is compatible with a member of $D \cap M$. We may assume that already $p' \in D$ (or else extend p' into D). The following lemma concludes the argument (apply it to p').

Lemma 2.2 *Suppose that $p = (x_p, d_p, N^p)$ is a condition in P (where $N^p = (N_0^p, \ldots, N_{k-1}^p)$), and $M \prec H(\kappa)$ is a countable elementary substructure such that $M \cap H(\aleph_2) = N_i^p$ for some $i < k$. Suppose moreover that $D \in M$ is dense open in P, and $p \in D$. Then p is compatible with a condition in $D \cap M$.*

Proof. Recycling the name p_0, define the "M-lower part" of p:

$$p_0 = p \restriction M = (x_p \cap M, d_p \cap M, \{N_0^p, \dots, N_{i-1}^p\})$$

as that part of p obtained by restriction to M. So $p_0 \in M \cap P$ since $N_{i-1}^p \in N_i^p$ and $N_i^p \subseteq M$. We shall define an extension q of p_0 in $D \cap M$ and prove that it is compatible with p. Of course, since $D \in M$ is dense, there is an extension of p_0 in $D \cap M$. The only possible obstacle is that this extension is not compatible with the rest of p (in case A_α intersects $x_q \setminus x_{p_0}$ for some $\alpha \in d_p$ not in M).

Suppose that $x_p = \alpha_0 < \alpha_1 < \cdots < \alpha_k$ where $\alpha_0, \dots, \alpha_i \in M$ but $\alpha_{i+1}, \dots, \alpha_k \notin M$. So $x_{p_0} = \alpha_0 < \cdots < \alpha_i$. Define

$$E = \{x \in [\omega_1]^k \mid x \text{ end-extends } x_{p_0} \text{ and for some } d, N : (x, d, N) \in D \text{ extends } p_0\}.$$

(If $x = (x_0, \dots, x_k)$ is an increasing enumeration, then x end-extends x_{p_0} means that $x_j = \alpha_j$ for $j \le i$.) E is non-empty since $x_p \in E$. Clearly $E \in M \cap H_{\aleph_2}$ because it is definable from parameters in M.

By downwards induction on $\ell = k, k-1, \dots, i$ define the formula $\varphi_\ell(b_{i+1}, \dots, b_\ell)$, where $b_{i+1} < \cdots < b_\ell$ are ordinal variables, which says the following:

(1) For $\ell = k$, $\varphi_k(b_{i+1}, \dots, b_k)$ just says that $(\alpha_0, \dots, \alpha_i, b_{i+1}, \dots, b_k) \in E$.

(2) For $i \le \ell < k$, $\varphi_\ell(b_{i+1}, \dots, b_\ell)$ says that there exists $Y_{\ell+1}$ not inside I such that for every $a_{\ell+1} \in Y_{\ell+1}$ $\varphi_{\ell+1}(b_{i+1}, \dots, b_\ell, a_{\ell+1})$.

In a more direct expression, we have for $\ell < k$ that $\varphi_\ell(b_{i+1}, \dots, b_\ell)$ says the following:

there exists $Y_{\ell+1}$ not inside I such that for every $a_{\ell+1} \in Y_{\ell+1}$
there exists $Y_{\ell+2}$ not inside I such that for every $a_{\ell+2} \in Y_{\ell+2}$

$\vdots$

there exists Y_k not inside I such that for every $a_k \in Y_k$

$$(\alpha_0, \cdots, \alpha_i, b_{i+1}, \cdots, b_\ell, a_{\ell+1}, \cdots, a_k) \in E.$$

For example, for $\ell = k-1$, $\varphi_{k-1}(b_{i+1}, \dots, b_{k-1})$ says: there exists Y_k not inside I such that for every $a_k \in Y_k$, $(\alpha_0, \cdots, \alpha_i, b_{i+1}, \cdots, b_{k-1}, a_k) \in E$.

For $\ell = i$, $\varphi_i()$ is the sentence which we want to get:
there exists Y_{i+1} not inside I such that for every $a_{i+1} \in Y_{i+1}$
there exists Y_{i+2} not inside I such that for every $a_{i+2} \in Y_{i+2}$

$\vdots$

there exists Y_k not inside I such that for every $a_k \in Y_k$
$(\alpha_0, \cdots, \alpha_i, a_{i+1}, \cdots, a_k) \in E$.

Claim 2.3 *For every $i \le \ell \le k$, $\varphi_\ell(\alpha_{i+1}, \dots, \alpha_\ell)$. In particular $\varphi_i()$.*

Proof. For $\ell = k$, the claim just says that $(\alpha_0, \dots, \alpha_k) \in E$. For the proof of the claim for $\ell < k$, assume that $\varphi_{\ell+1}(\alpha_{i+1}, \dots, \alpha_{\ell+1})$ holds, and then use the following lemma.

Lemma 2.4 *If $N \prec H_{\aleph_2}$ is countable, and $b \in \omega_1 \setminus N$ is not member of any set in N that is inside I, if $a \in N$ and $\phi(x, y)$ is any formula such that $\phi(a, b)$ holds, then $Y = \{y \in \omega_1 \mid \phi(a, y)\}$ is not inside I.*

Proof. Since Y is definable in N it belongs to N, and as it contains b it cannot be inside I. □

Now we show how to get an extension of p_0 in $D \cap M$ that is compatible with p and thus conclude the proof of Lemma 2.2. Let $X = \bigcup\{A_\alpha \mid \alpha \in d_p\}$. Then $X \in I$. Using $\varphi_i()$ we get $Y_{i+1} \in M$ not inside I and we can pick $a_{i+1} \in Y_{i+1} \cap M \setminus X$. (In details the argument goes as follows. Since Y_{i+1} is not inside I it contains a countable subset Y'_{i+1} not in I, and we may take it in M. Since it is a countable set, we have $Y'_{i+1} \subset M$ and since $Y'_{i+1} \not\subset X$ we can get a_{i+1} as required.) Then we get $Y_{i+2} \in M$ not inside I, and we can pick $a_{i+2} \in Y_{i+2} \cap M \setminus X$ by the same argument. We continue in this way until we finish with some $a_k \notin X$ so that $(\alpha_0, \cdots, \alpha_i, a_{i+1}, \cdots, a_k) \in E \cap M$. By definition of E we get an extension q of p_0 in $D \cap M$ of the form $q = (\langle \alpha_0, \cdots, \alpha_i, a_{i+1}, \cdots, a_k \rangle, d, N)$. The choice of the a_j's out of X ensures that q and p are compatible. □

Second Form. The second form of the dichotomy is the following.

> If I is any ω_1 generated ideal over a set S then: Either there is a decomposition $S = \bigcup_{i \in \omega} S_i$ where each S_i is out of I, or there is an uncountable subset of S that is inside I.

Theorem 2.5 *This second form is a consequence of Martin's Axiom $+2^{\aleph_0} > \aleph_1$ and the first form of the dichotomy. Hence it holds under the PFA.*

Proof. Assume that I is an ω_1 generated ideal of countable sets, so that no uncountable subset of ω_1 is inside of I. We may assume that I consists of countable subsets of ω_1. We shall find a decomposition of ω_1 into countably many sets that are out of I (orthogonal to I). If $X \subseteq \omega_1$ is any uncountable subset, then since X does not contain an uncountable subset that is inside I, it surely is not the countable union of sets that are inside I. Hence by the First Form applied to X, X contains an uncountable subset that is outside of I. Thus, every uncountable subset of ω_1 contains an uncountable subset that is out of I. Now we shall see that a c.c.c forcing poset introduces a partition $\omega_1 = \bigcup_{i \in \omega} S_i$ where every S_i is out of I.

Theorem 2.6 *Let I be an ω_1 generated ideal of countable subsets of ω_1 such that every uncountable subset of ω_1 contains an uncountable subset that is out of I. Then there is a c.c.c poset that forces a partition of ω_1 into countably many subsets that are out of I.*

Proof. Say $\{A_\alpha \mid \alpha < \omega_1\}$ generate the ideal I (with $A_\alpha \subseteq \alpha$). Define P as the collection of all pairs $p = (f_p, d_p)$ where f_p is a finite function from ω_1 to ω, and d_p is a finite subset of ω_1. The intuition is that f_p will develop generically as a function f_G that partitions ω_1 into countably many subsets out of I (namely the sets $f_G^{-1}\{n\}$), and the role of d_p is to "freeze" the finite intersection of $f_p^{-1}\{n\}$ with A_α for $\alpha \in d_p$.

Accordingly, we define $q \le p$ iff f_q and d_q extend f_p and d_p, and, for every $\alpha \in d_p$, for every k in the range of f_p, $f_p^{-1}\{k\} \cap A_\alpha = f_q^{-1}\{k\} \cap A_\alpha$.

Note that in order for p and q to be compatible it is necessary and sufficient that (1) $f_p \cup f_q$ is a function, (2) that for every k in the range of f_p, and $\alpha \in d_p$, for every $x \in f_q^{-1}\{k\} \setminus f_p^{-1}\{k\} : x \notin A_\alpha$, and (3) the same with q and p interchanged.

Simple arguments show that any condition p can be extended to p' which includes in the domain of $f_{p'}$ an arbitrary countable ordinal. Likewise, d_p can be extended.

If we let G be a generic filter over P, then $f_G : \omega_1 \to \omega$, and $\omega_1 = \bigcup\{d_p \mid p \in G\}$. It is clear that for every $n \in \omega$ and $\alpha < \omega_1$ $(f_G^{-1}\{n\}) \cap A_\alpha$ is finite. So that $f_G^{-1}\{n\}$ is indeed out of I. It remains to prove that P is c.c.c.

For this aim, let $\{p_\alpha \mid \alpha < \omega_1\}$ be given, with $p_\alpha = (f_\alpha, d_\alpha)$. Assume that the domains of f_α and the sets d_α form Δ systems. We may assume that the core of each of these two Δ-systems is empty. That is, the functions f_α have disjoint domains, and the sets d_α are pairwise disjoint. (For if we remove the core from the conditions and find two that are compatible, then the original conditions are compatible as well.) Say all f_α have the same cardinality k, and the domain of f_α is notated $a_\alpha^0, \dots, a_\alpha^{k-1}$ in increasing order. We may assume that if p_β is in the Δ system, then no ordinal in $\mathrm{dom}(p_\beta) = \mathrm{dom}(f_\beta) \cup d_\beta$ is below β (by Fodor's and as the cores are empty). We may also assume that if both p_α and p_β are in the Δ-system (and $\alpha < \beta$) then all ordinals in p_α are below β (and hence disjoint to $\mathrm{dom} p_\beta$). Define $A(p_\alpha) = \bigcup\{A_\xi \mid \xi \in d_\alpha\}$. Since $A_\xi \subseteq \xi$, condition $d_\alpha \subset \beta$ implies $A(p_\alpha) \subset \beta$. Hence the only obstacle for p_α and p_β to be compatible (where $\alpha < \beta$) is that some a_α^i is in $A(p_\beta)$.

If p_α and p_β are both in the Δ-system for $\alpha < \beta$ then $p_\alpha \cup p_\beta$ is a condition that extends both p_α and p_β if $\mathrm{dom}(p_\alpha) \cap A(p_\beta) = \emptyset$. We will see how to get this situation, which provides two compatible conditions.

Applying k times the property that every uncountable subset of ω_1 contains an uncountable subset that is out of I, we obtain uncountable $E \subset \omega_1$ such that for every $j < k$ the set formed by the jth components $\{a_\alpha^j \mid \alpha \in E\}$ is an uncountable set out of I. Take $E_0 \subset E$ be countable infinite, and let p_β be with β above all ordinals of conditions in p_α for $\alpha \in E_0$. For every $j < k$, the intersection of $A(p_\beta)$ with $\{a_\alpha^j \mid \alpha \in E_0\}$ is finite, and so we can find some $\alpha \in E_0$ such that $A(p_\beta)$ has no intersection with $\{a_\alpha^j \mid j < k\}$. So p_β and p_α are compatible. □

Here is an application due to Todorcevic of the Symmetric Dichotomy theorem.

Theorem 2.7 *PFA implies that there no S-spaces. In fact, the simple symmetric dichotomy for $\aleph_1$-generated ideals implies that there are no S-spaces.*

Proof. Recall the definition: An S-space is a regular, hereditarily separable, but not hereditarily Lindelof topological space.[2] To prove that no such space exists (under the dichotomy), suppose that X is a regular topological space which is not hereditarily

[2] Hereditarily separable–every subspace has a countable dense subset. Hereditarily Lindelof–every cover of a subspace has a countable subcover.

Lindelof and we shall prove that X is not hereditarily separable. Since X is not hereditarily Lindelof, X has a subspace $S = \{x_\alpha \mid \alpha < \omega_1\}$ such that every initial part $S_\delta = \{x_\alpha \mid \alpha \leq \delta\}$ is open in S (i.e. S is "right-separated"). We consider the subspace topology on S and shall find a subset of S which is not separable.

Since S is regular, each x_α has an open neighborhood U_α with closure $\overline{U}_\alpha \subset S_\alpha$. These countable closed sets generate an ideal I. By the symmetric dichotomy, there is an uncountable set $D \subset S$ which is either "inside" or "outside" of I. If D is inside, then every countable subset E of D is in I, which means that it is covered by a countable closed set, and hence E is not dense in D. If D is outside of I, then D has a finite intersection with every set in I. So in particular the intersection of D with every U_α is finite. As S is a Hausdorff space, D is discrete (and therefore not separable). □

PID: the P-ideal dichotomy

We say that $I \subseteq [S]^{\leq\omega}$ is a P-ideal if it is an ideal (containing all singletons of S) such that whenever $X_n \in I$ for $n \in \omega$ then there is $X \in I$ such that $X_n \subseteq^* X$ for all n. We say that X is an "almost cover" for $\{X_n \mid n \in \omega\}$. ($A \subseteq^* B$ is almost inclusion, which means that $A \setminus B$ is finite.)

It turns out that for P-ideals the dichotomy can be obtained consistently with CH ([1]), and the restriction that the ideal is ω_1 generated can be removed ([3]).

If I is an ideal over S and $S_0 \subset S$, then the restriction of I on S_0 is the ideal $\{X \cap S_0 \mid X \in I\}$. If I is a P-ideal, then its restriction is again a P-ideal.

The following theorem shows that the dichotomy of Form 2 holds for any P-ideal I over S assuming the PFA.

Theorem 2.8 *Assume PFA. Let I be a P-ideal over an arbitrary (uncountable) set S. Either there is a decomposition $S = \bigcup_{i\in\omega} S_i$ where each S_i is out of I, or there is an uncountable subset of S that is inside I. This property is known as the PID.*

The proof is by induction on the cardinality of S. So assume that $|S| = \mu$ and that the dichotomy holds for any P-ideal over a set of cardinality smaller than μ. Suppose that S is not a countable union of sets that are out of I and we shall find an uncountable set that is inside I. If an uncountable $S_0 \subseteq S$ of cardinality $< \mu$ is not a countable union of sets that are out of I then by the inductive hypothesis there is an uncountable set inside of I and we are done. Hence we may assume that every uncountable subset of S of smaller cardinality is indeed a countable union of sets out of I. We may assume $S = \mu$; that is, the ideal is over the cardinal μ itself. The minimality of μ implies that its cofinality is $> \omega$

We say that $K \subseteq I$ is cofinal in I if it is cofinal in the almost inclusion ordering $\subseteq^*$. That is, for every $X \in I$ there is $Y \in K$ such that $X \subseteq^* Y$.

The following lemma will be needed later on.

Lemma 2.9 *Suppose I is a P-ideal and $A \in I$ is (countable) infinite, and for every $a \in A$ we have some $X(a) \in I$. If K is cofinal in I, then there are $Y \in K$ and $a \in A$ such that*

$$X(a) \subseteq^* Y \text{ and } a \in Y.$$

Proof. Since I is a P-ideal, there is some $Z \in I$ such that $A \cup \bigcup\{X(a) \mid a \in A\} \subseteq^* Z$. As K is cofinal, there is some $Y \in K$ such that $Z \subseteq^* Y$. Since A is infinite, we can find some $a \in A \cap Y$. We also get $X(a) \subseteq^* Y$ as required. □

Let P be the poset of all pairs $p = (a_p, H_p)$ where $a_p \in I$ (so a_p is countable), and H_p is a countable collection of cofinal subsets of I. Define $q \leq p$ iff a_q is an end-extension of a_p (that is a_p is an initial segment of a_q)[3] $H_p \subseteq H_q$, and the following condition holds.

For every $K \in H_p$, if $e = a_q \setminus a_p$ then $\delta(e, K) = \{X \in K \mid e \subseteq X\} \in H_q$.

Note that requiring $\delta(e, K) \in H_q$ implies in particular that $\delta(e, K) \subseteq K$ is cofinal in I.

Note that $\leq$ is indeed transitive.

The role of a_p is to give a finite information on the generic uncountable set inside I. The sets of H_p help to ensure properness of P under the Machiavellian advice "adjoin your enemy into your court".

Let $p = (a_p, H_p)$ be a condition. If e is any finite set disjoint from a_p, define $p + e$ as the pair $(a_p \cup e, H')$ where $H' = H \cup \{\delta(e, K) \mid K \in H_p\}$. Clearly $p+e$ is a condition iff each $\delta(e, K)$ for $K \in H_p$ is cofinal in I. In case $p+e$ is a condition, it is an extension of p.

If p is any condition then a *pre-extension of p* is a pair (a, H) such that $a_p \subseteq a$, $H_p \subseteq H$ is a countable collection of cofinal in I sets, and for every $K \in H_p$, $\delta(a \setminus a_p, K)$ is cofinal in I (but is not necessarily in H). If (a, H) is a pre-extension of p, then (a, H') is an extension of p, where $H' = H \cup \{\delta(a \setminus a_p, K) \mid K \in H_p\}$.

Lemma 2.10 *Assume S is not a countable union of sets out of I but every subset of S of smaller cardinality is such a union. Then every condition $p \in P$ has, for any ordinal $\gamma < \mu = S$, an extension q so that a_q contains an ordinal $\geq \gamma$.*

To prove the lemma suppose on the contrary that every $\alpha \geq \gamma$ cannot be added to a_p (i. e. $p + \{\alpha\}$ is not a condition). Then there is a reason $K(\alpha) \in H_p$ such that $\delta(\{\alpha\}, K(\alpha))$ is not cofinal. Thus there exists some $X(\alpha) \in I$ so that no set in $\delta(\{\alpha\}, K(\alpha))$ almost includes $X(\alpha)$. That is, no set in $K(\alpha)$ that contains α also almost includes $X(\alpha)$.

Since H_p is countable, this yields a decomposition of $\mu \setminus \gamma$ into countably many classes, namely for every $K \in H_p$ we have the class C_K of those α such that $K = K(\alpha)$. But C_K is out of I (by Lemma 2.9), and the ordinals below γ have a countable decomposition into out-of-I sets by the assumption. So μ is a countable decomposition into out-of-I sets which is a contradiction. □

The main point in the proof of Theorem 2.8 is to prove that P is proper. For this let M be countable elementary substructure of some large $H(\kappa)$, and suppose that

[3] Requesting only that $a_p \subseteq a_q$ would also work, but is slightly more cumbersome.

$p_0 \in P \cap M$ is given. Let $\{D_i \mid i < \omega\}$ be a countable enumeration of all dense sets that are in M. Find $X_M \in I$ that almost covers every set in $I \cap M$ (I is a P-ideal, and $I \cap M$ is a countable subset of I). We may require that $a_0 = a_{p_0} \subset X$. Starting with p_0, we are going to define an increasing sequence $p_i = (a_i, H_i) \in P \cap M$ with $p_{i+1} \in D_i$. The main additional requirement is that $a_i \subset X_M$. We shall finally define $p_\omega = (a_\omega, H_\omega) = (\bigcup_{i\in\omega} a_i, \bigcup_{i\in\omega} H_i)$ aiming that this condition is a pre-extension of every p_i. The fact that $a_i \subset X_M$ ensures that $\bigcup_{i\in\omega} a_i \in I$. This is the main difficulty of the construction, to get that $a_i \subset X_M$ and at the same time to ensure that the sequence is M-generic. It is obvious that any p_i can be extended into a condition $p_{i+1} \in D_i \cap P$, and clearly $a_{i+1} \subset^* X_M$, but getting $a_{i+1} \subset X_M$ is the problem.

An additional difficulty is that we cannot expect that p_ω is an extension of each p_j, for if we let $e_j = a_\omega \setminus a_j$, then there is no reason to have $\delta(e_j, K) \in H_\omega$ for $K \in H_j$. We will ensure however that p_ω is a pre-extension of every p_j by reducing X_M at each step of the inductive definition of the p_j sequence.

The following lemma will help us.

Lemma 2.11 *Suppose $p = (a_p, H_p) \in P \cap M$ is an arbitrary condition, $D \in M$ is dense in P, and X almost covers every member of $I \cap M$. Then some extension q of p in $D \cap M$ is such that $a_q \setminus a_p \subset X$.*

Proof. We say that $Y \in I$ is "bad" iff for some finite $F \subset Y$ for every extension $q \in D$ of p, $a_q \setminus a_p \not\subseteq Y \setminus F$. We claim that some $Y \in I$ is not bad. Suppose on the contrary that every $Y \in I$ is bad and some finite $F_Y \subset Y$ is the evidence. Now the collection $L = \{Y \setminus F_Y \mid Y \in I\}$ is trivially cofinal in I (in fact $Y \subseteq^* Y \setminus F_Y$). Let $p_1 = (a_p, H_p \cup \{L\})$ be the extension of p_0 obtained by adding L. Find $q \in D$ that extends p_1. Then, by definition of extension there exists some $A \in L$ such that $a_q \setminus a_p \subseteq A$ (in fact a cofinal set of such A). Now, $A = Y \setminus F_Y$ for some $Y \in I$, and this is a contradiction.

Thus some $Y \in I$ is not bad, and we can take it to be in M (which is an elementary substructure). Hence $Y \subseteq^* X$. Say $F = Y \setminus X$. Then F is finite, and as Y is not bad there exists some $q \in D$ extending p and such that $a_q \setminus a_p \subseteq Y \setminus F \subseteq X$.

□-Lemma 2.11

Using this lemma it is possible to get an increasing M-generic sequence of conditions so that $\bigcup_{i\in\omega} a_i$ is a subset of X and is hence in I. But this is not enough and there is an additional problem. To obtain that p_ω is a pre-extension of p_i, we require that

$$\text{for every } K \in H_i : \ \delta(e_i, K) \text{ is cofinal in } I, \tag{1}$$

where $e_i = (\bigcup_{j\in\omega} a_j) \setminus a_i$. For this aim, we shall also define a sequence $X^i_M \in I$, $X^{i+1}_M \subseteq X^i_M$ of almost covers of $I \cap M$, and at the i-th stage we will get p_{i+1} so that

$$a_{i+1} \setminus a_i \subset X^i_M. \tag{2}$$

As we shall explain, a certain choice of X^i_M is going to assure that (1) will hold. For this strategy to work, we need another lemma.

Lemma 2.12 *Suppose $X \in I$ and $L \subseteq I$ is cofinal. Then for some finite $F \subset X$, $\delta(X \setminus F, L)$ is cofinal.*

Proof of the lemma. Suppose that this is not true and for every finite $F \subset X$ the set $\delta(X \setminus F, L) = \{A \in L \mid X \setminus F \subseteq A\}$ is not cofinal. So let $Y(F) \in I$ be such that there is no $A \in L$ with $X \setminus F \subseteq A$ that almost covers $Y(F)$. Since I is a P-ideal, there is a set A in I that almost covers X and each of the $Y(F)$ sets for $F \in [X]^{<\omega}$. As L is cofinal in I, we can take $A \in L$. But now $F = X \setminus A$ yields a contradiction. □-Lemma 2.12

Now the construction of the M-generic sequence $p_i \in P \cap M$ can be described with more details. We have an enumeration of all dense subsets of P that are in M, and we require in defining P_{j+1} that we enter the jth dense set D_j in this enumeration. But we have (1) as an additional mission. Since every H_i is countable, we can fix an enumeration of H_i, and so for every $K \in H_i$ there is a stage $j \geq i$ so that in defining p_{j+1} (assuming p_j is already defined) we are required to take care of K. Since $p_i \leq p_j$, $L = \delta(a_j \setminus a_i, K) \in H_j$ is one of the cofinal sets there. Applying Lemma 2.12 to $X^j_M \in I$ and L, we get a finite $F \subset X^j_M$ such that

$$\{A \in L \mid X^j_M \setminus F \subseteq A\} \text{ is cofinal in } I.$$

So if we define $X^{j+1}_M = X^j_M \setminus F$ then (2) ensures that for every $k > j$ $a_k \setminus a_j \subseteq X^{j+1}_M$, and so $e_j = (\bigcup_{k>j} a_k) \setminus a_j \subseteq X^{j+1}_M$. Hence $\delta(e_j, L) \supseteq \delta(X^{j+1}_M, L)$, and so $\delta(e_j, L)$ is cofinal in I. But $\delta(e_i, K) = \delta(e_j, L)$ and hence $\delta(e_i, K)$ will be cofinal. □ Note that our proof of properness shows that P adds no new countable sets (of ordinals).

Our aim now is to apply the PFA in order to obtain an uncountable subset of μ that is inside I. Let D_γ for $\gamma < \omega_1$ be the set of conditions p with a_p of order-type $\geq \gamma$. Then D_γ is dense in P. This can be proved by induction on γ. For successor ordinals, use Lemma 2.10. For limit γ, take a countable elementary substructure M as in the properness proof with $\gamma \in M$ and use the assumption that for every $\gamma' < \gamma$ the corresponding set $D_{\gamma'}$ is dense to deduce that the generic condition finally obtained has a_ω of order-type $\geq \gamma$.

Finally, using the PFA we can get a filter G over P that intersects every D_γ and hence $a = \bigcup_{p \in G} a_p$ has order-type ω_1 and any countable subset of a is a subset of some a_p and is hence in I.

3. Application of PID: no squares

Assuming PID Todorcevic proved that every coherent sequence is threadable, and hence there are no square sequences. This result is based on Todorcevic analysis of walks and so we begin with this.

3.1 Walks

A club system on a limit ordinal λ with uncountable cofinality is a sequence $C = \langle C_\alpha \mid \alpha \in \lambda \rangle$ such that for limit $\alpha < \lambda$ C_α is a club (closed unbounded) subset of α, and $C_\alpha = \{\beta\}$ when $\alpha = \beta + 1$. It is convenient to assume that 0 is always in C_α.

We consider the following properties a club system may have:

(1) C is coherent if whenever $\alpha \in \lim C_\beta$ we have $C_\alpha = C_\beta \cap \alpha$. (For any set of ordinals C, $\lim C$ is the set of those limit $\delta \in C$ for which $\delta \cap C$ is unbounded in δ.)

(2) C is "threadable" (or trivial) iff it can be extended to a $\lambda + 1$ coherent system. That is, there is a club C_λ in λ such that for every $\delta \in \lim C_\lambda$, $C_\delta = C_\lambda \cap \delta$.

(3) Jensen's $\square_\kappa$ sequence for a cardinal κ is a coherent club sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ such that the order-type of each C_α is $\leq \kappa$.

Definition of walks. Let $\langle C_\alpha \mid \alpha < \lambda \rangle$ be a fixed club system on some ordinal λ with uncountable cofinality. For every $\alpha \leq \beta < \lambda$ we shall define the walk from β down to α $walk(\alpha,\beta) = \langle \beta_0, \ldots, \beta_{n-1} \rangle$ so that $\beta_0 > \beta_1 \ldots$ is descending, starting with $\beta_0 = \beta$ and ending with $\beta_{n-1} = \alpha$. The definition is by induction on β, and then we also set $\rho_2(\alpha,\beta) = n - 1$ (Since we have here just one ρ function we shall write $\rho(\alpha,\beta)$ rather than $\rho_2(\alpha,\beta)$.)

$$walk(\alpha, \alpha) = \langle \alpha \rangle$$

Correspondingly $\rho(\alpha, \alpha) = 0$. For $\beta > \alpha$ we define:

$$walk(\alpha,\beta) = \langle \beta \rangle^\frown walk(\alpha, \min(C_\beta \setminus \alpha)).$$

Correspondingly $\rho(\alpha,\beta) = 1 + \rho(\alpha, \min(C_\beta \setminus \alpha))$. Note that $C_\beta \setminus \alpha \neq \emptyset$ and hence $\min(C_\beta \setminus \alpha)$ is meaningful.

Although the inductive definitions of *walk* and ρ facilitate proofs, an intuitive description is also important. The definition of the ordinals $\beta_i \geq \alpha$ that constitute the walk from β down to α is by the following procedure. We start with $\beta_0 = \beta$. Suppose that $\beta_0 > \cdots > \beta_i$ have been defined. If $\beta_i = \alpha$ the procedure stops, but if $\beta_i > \alpha$ let β_{i+1} be the first ordinal in C_{β_i} that is not below α (there is such an ordinal since C_{β_i} is unbounded in β_i). In case β_i is a successor ordinal, β_{i+1} is the predecessor of β_i. The procedure must stop since there is no infinite descending sequence of ordinals, and hence we must arrive to some $\beta_i = \alpha$. The ordinals β_i are said to be "on the walk".

If $walk(\alpha,\beta) = \langle \beta_0, \ldots, \beta_{n-1} \rangle$ then (β_0, β_1) is the "first step", (β_1, β_2) the second etc. So $\rho(\alpha,\beta)$ is the number of steps in the walk.

Lemma 3.1 *If the club system $\langle C_\alpha \mid \alpha < \lambda \rangle$ is coherent, then for every $\alpha < \beta < \lambda$*

$$\sup_{\xi < \alpha} |\rho(\xi, \alpha) - \rho(\xi, \beta)| < \infty. \tag{3}$$

In other words, there is a bound $k < \omega$ that depends only on α and β so that for every $\xi < \alpha$ the walking distances between α to ξ and β to ξ do not differ by more than k.

Proof. By induction on β. There are two simple cases and one that needs more attention.

(1) If $\alpha \in \lim C_\beta$ then $C_\alpha = C_\beta \cap \alpha$ (by coherence) and, for every $\xi < \alpha$, $walk(\xi,\beta) = walk(\alpha,\beta)$. Hence $\rho(\xi,\beta) = \rho(\xi,\alpha)$. This is a simple case that does not rely on the inductive assumption.

(2) If $\beta = \beta_0 + 1$ is a successor, then $\rho(\xi,\beta) = 1 + \rho(\xi,\beta_0)$, and (3) is trivial in this case by the inductive assumption for β_0.

So assume that β is a limit ordinal and $\alpha \notin \lim C_\beta$. Let $\beta_0 < \alpha$ be $\max \alpha \cap \lim C_\beta$ (there is a max since $\alpha \notin \lim C_\beta$ and $0 \in \lim C_\beta$). Then $C_\beta \cap [\beta_0, \alpha) = \{\beta_0, \ldots, \beta_{k-1}\}$ is finite ($\beta_0 < \beta_1 < \cdots < \beta_{k-1}$). Say $\beta_k = \min(C_\beta \setminus \alpha)$. For $\xi < \alpha$ there is a finite number of possibilities as to the relation of ξ with the β_i's, and so it suffices to prove the lemma for each of these possibility types. If $\beta_i \le \xi < \beta_{i+1}$, then the first step in $walk(\xi,\beta)$ is into β_{i+1}, and hence $\rho(\xi,\beta) - \rho(\xi,\beta_{i+1}) = 1$ on this interval. Yet $\rho(\xi,\beta_{i+1}) - \rho(\xi,\alpha)$ is bounded by the inductive assumption. Hence the differences between $\rho(\xi,\alpha)$ and $\rho(\xi,\beta)$ are bounded on this interval. A similar argument is for the other cases. □

Lemma 3.2 *If $C = \langle C_\alpha \mid \alpha \in \lambda\rangle$ is coherent (where λ is any ordinal with uncountable cofinality) then C is threadable iff for some $A, B \subseteq \lambda$ both unbounded in λ there is $n \in \omega$ such that*

$$\text{for all } \alpha \in A,\ \beta \in B, \text{ such that } \alpha < \beta, \text{ we have } \rho(\alpha,\beta) \le n. \tag{4}$$

Proof. Suppose that C is threadable, and $D \subseteq \lambda$ is the club such that $C_\alpha = D \cap \alpha$ for all $\alpha \in \lim D$. Then $A = B = D' = \lim D$ works with $n = 1$. Indeed if $\alpha < \beta$ are in D' then $C_\alpha = C_\beta \cap \alpha$. So $\rho(\alpha,\beta) = 1$.

For the other direction, let $n \ge 1$ be minimal such that for some unbounded $A, B \subseteq$ $\subseteq \lambda$ (4) holds.

Let S be the stationary subset of λ of limit ordinals. Define

$$X = \{\delta \in S \mid \forall \beta \in B \setminus (\delta + 1)\ \delta \in \lim C_\beta\}.$$

If X is unbounded in λ then $D = \bigcup_{\delta\in X} C_\delta$ is a thread. (The main point is that if $\delta_1 < \delta_2$ are both in X then there exists β (any $\beta \in B$ above δ_2 will do) so that $\delta_1, \delta_2 \in \lim C_\beta$. So $C_{\delta_1} = C_\beta \cap \delta_1$ and $C_{\delta_2} = C_\beta \cap \delta_2$, and hence C_{δ_2} is an end-extension of C_{δ_1}.) So assume that X is bounded by $\gamma_0 < \lambda$ and we shall get a contradiction to the minimality of n.

For every $\delta \in S \setminus (\gamma_0 + 1)$ pick an arbitrary $\beta(\delta) \in B$ such that $\beta(\delta) > \delta$ and $\delta \notin \lim C_{\beta(\delta)}$, and then let $\gamma(\delta) < \delta$ be a bound of $C_{\beta(\delta)} \cap \delta$, and let $b(\delta) = \min C_{\beta(\delta)} \setminus \delta$. So we have $\gamma(\delta) < \delta \le b(\delta) < \beta(\delta) \in B$. By Fodor's lemma applied to the function $\delta \mapsto \gamma(\delta)$, we get a stationary $S_1 \subset S$ and some fixed γ_1 so that $\gamma(\delta) < \gamma_1$ for all $\delta \in S_1$. Keep in mind that the interval $(\gamma_1, b(\delta))$ has an empty intersection with $C_{\beta(\delta)}$ for $\delta \in S_1$.

Now observe that if $\alpha \in A$ is above γ_1 and $b(\delta) > \alpha$ (is with an arbitrary $\delta \in S_1$) then $\rho(\alpha, b(\delta) \le n - 1$. Why? Because we have that $\alpha < b(\delta) < \beta(\delta)$ and $\beta(\delta) \in B$. So

$\rho(\alpha,\beta(\delta)) \le n$. But the first step in $walk(\alpha,\beta(\delta))$ is onto $b(\delta)$. And hence $walk(\alpha,b(\delta))$ contains one step less than $walk(\alpha,\beta(\delta))$.

So, by replacing B with the unbounded set $\{b(\delta) \mid \delta \in S\}$ and replacing A with $A \setminus \gamma_1$ we get a contradiction to the minimality of n. □

For every coherent sequence $C = \langle C_\alpha \mid \alpha \in \lambda \rangle$ (with λ of uncountable cofinality), Todorcevic defined the corresponding ideal I_C as the collection of all countable subsets $X \subset \lambda$ such that for some $\beta \ge \sup X$ we have that $\lim_{x \in X} \rho(x,\beta) = \infty$ (by this we mean that for every $n \in \omega$, for all but finitely many $x \in X$ we have $\rho(x,\beta) > n$. So a finite set is trivially in the ideal.) Another way to formulate this property of β, is to say that the function $\lambda x\, \rho(x,\beta)$ is finite-to-one on X. Lemma 3.1 implies that if $X \in I_C$ then actually for *every* $\beta \ge \sup X$ we have that $\rho(x,\beta)$ tends to infinity as $x \in X$.

Lemma 3.3 *I_C is a P-ideal.*

Proof. Suppose $X_i \in I_C$ for $i \in \omega$. We shall find $X \in I_C$ such that $X_i \subseteq^* X$ for all $i \in \omega$. As $cf\lambda > \omega$, $\sup \bigcup_{i \in \omega} X_i < \lambda$, and we can pick β above that sup. Let X'_i be obtained from X_i by removing all $\alpha \in X_i$ such that $\rho(\alpha,\beta) \le i$ (a finite number of them). Define $X = \bigcup_{i \in \omega} X'_i$. □

Assuming the P-ideal dichotomy there are two alternatives: 1) there is a decomposition of λ into countably many sets out of the ideal, or 2) there is an uncountable set inside the ideal.

Theorem 3.4 (Todorcevic) *Assuming PID, every coherent sequence over λ where* $\mathrm{cf}(\lambda) > \omega_1$ *is threadable.*

Proof. Let C be a coherent sequence over λ. Then $I = I_C$ is a P-ideal, and we rule out the second possible consequence of PID. Namely, we rule out the possibility that $A \subset \lambda$ is uncountable and inside I. We may assume that A has order-type ω_1 and is thence bounded in λ; so there is $\beta \in \lambda > \sup A$ (as $\mathrm{cf}(\lambda) > \omega_1$). Supposedly we have $\lim_{x \in X_0} \rho(s,\beta) = \infty$ for every countable $X_0 \subset A$, but this is impossible since there has to be an infinite subset of A over which $\lambda x \rho(x,\beta)$ is constant.

Hence by the PID there is $A \subseteq \lambda$ unbounded and out of I (if there is a decomposition into countably many orthogonal sets then one such set U is unbounded). For every $\beta < \lambda$ there is no infinite sequence $x_n \in U \cap \beta$ with $\lim_n \rho(x_n,\beta) = \infty$ (or else $\{x_n \mid n \in \omega\}$ would be an infinite subset of U in I). So for every β there is some $n(\beta)$ so that if $x \in U \cap \beta$ then $\rho(x,\beta) \le n(\beta)$. Then $n = n(\beta)$ for an unbounded set B of such βs. Then Theorem 3.2 shows that C is threadable. □

4. $\mathfrak{b} \le \omega_2$

There is a conjecture that PID implies that $\mathfrak{c} \le \aleph_2$. Todorcevic has proved that PID implies $\mathfrak{b} \le \omega_2$, where $\mathfrak{b}$ is the least cardinality of an unbounded subset of ω^ω in the almost everywhere dominance relation $<^*$. We describe this proof here.

For $f, g \in \omega^\omega$ we write $f <_n g$ iff $\forall k \geq n(f(k) < g(k))$. Eventual dominance is written $f <^* g$ (which means $\exists n f <_n g$). If $f <^* g$ let $\chi(f, g)$ be the least n such that $f <_n g$. Clearly, $g_1 \leq_0 g_2$ implies $\chi(f, g_1) \leq \chi(f, g_2)$.

Let $\langle f_\xi \mid \xi \in \kappa \rangle$ be a $<^*$-increasing sequence of functions, and suppose that $g \in \omega^\omega$ is an upper bound of that sequence. Then define $I_g \subset [\kappa]^{\leq \aleph_0}$ as follows: $X \in I_g$ iff for every $n \in \omega$, $\{\xi \in X \mid \chi(f_\xi, g) \leq n\}$ is finite.

A fuller notation for that ideal would be $I_{g, \langle f_\xi | \xi \in \kappa \rangle}$. Yet we plan to have $\langle f_\xi \mid \xi \in \kappa \rangle$ fixed, and hence the shorter notation. A simple argument shows:

Lemma 4.1 *I_g is a P-ideal.*

The following follows directly from the definition.

Lemma 4.2 *If both g_0 and g_1 are upper bounds of the increasing sequence $\langle f_\xi \mid \xi \in \kappa \rangle$, then $g_1 <^* g_0$ implies that $I_{g_0} \subseteq I_{g_1}$.*

For any $A \subseteq \kappa$, define $f_A = \sup \langle f_\xi \mid \xi \in \kappa \rangle$ as the following function $f_A \in (\omega \cup \{\infty\})^\omega$ defined by

$$f_A(n) = \sup\{f_\xi(n) \mid \xi \in A\}.$$

Lemma 4.3 *If $A \subseteq \kappa$ is out of the ideal I_g, then $f_A <^* g$. So f_A can hit ∞ only a finite number of times, and is hence an upper bound of $\langle f_\xi \mid \xi \in A \rangle$ that is $<^*$ below g.*

Proof. Assume that $\{n \in \omega \mid f_A(n) \geq g(n)\}$ is infinite. Then we can define $\xi(i) \in A$ for $i \in \omega$ such that $\chi(f_{\xi(i)}, g) \geq i$ for all i. Thus $A_0 = \{\xi(i) \mid i \in \omega\} \in I_g$ contradicts our assumption that A is out of I_g. □

Every uncountable subset of κ contains a countable set out of I_g, because if $X \subseteq \kappa$ is uncountable, then for some uncountable subset $X_0 \subseteq X$ and fixed n we have $\chi(f_\xi, g) = n$ for all $\xi \in X_0$. So no infinite subset of X_0 is in I_g. That is, X_0 is outside of I_g. Thus no uncountable subset of κ is inside I_g. If PID holds, then we must have that κ is a countable union of sets that are outside of I_g. If case $\mathrm{cf}(\kappa) > \omega$, we therefore find a set cofinal in κ that is outside of I_g.

Theorem 4.4 *PID implies $\mathfrak{b} \leq \omega_2$.*

Proof. Clearly $f <^* g$ implies that $g(k) = 0$ only a finite number of times. Hence if $\langle f_\xi \mid \xi < \kappa \rangle$ is an increasing (infinite) sequence in ω^ω dominated by g, then the function g^- defined by the equation $g^-(k) = g(k) \dot{-} 1$ dominates each f_ξ and is below g.

Lemma 4.5 *Assume $\kappa < \mathfrak{b}$. $\langle f_\xi \mid \xi < \kappa \rangle$ is an increasing sequence in the $<^*$ relation, and $\langle g_i \mid i \in \omega \rangle$ is a decreasing sequence of functions in that relation such that $f_\xi <^* g_i$ for every ξ and i. Then there exists $g \in \omega^\omega$ such that g dominates each f_ξ and $g <^* g_i$ for every $i \in \omega$.*

Proof. For every $\xi \in \kappa$ define $r_\xi \in \omega^\omega$ by $r_\xi(i) = \chi(f_\xi, g_i)$. Using assumption $\kappa < \mathfrak{b}$, let $r \in \omega^\omega$ be such that $r_\xi <^* r$ for every $\xi < \kappa$. We may assume that r is increasing ($r(i) < r(j)$ when $i < j$), $r(0) = 0$, and that $\chi(g_i, g_{i-1}) \leq r(i)$ for every $i > 0$. Define $g = \bigcup_{i \in \omega} g_i \restriction [r(i), r(i+1))$. We claim that $g <^* g_i$ for every i. In fact $g <_{r(i+1)} g_i$ can be easily proved. We also claim that $f_\xi <^* g$ for every $\xi \in \kappa$. As $r_\xi <^* r$, there is $n \in \omega$ such that for all $i \geq n$ $r_\xi(i) < r(i)$ holds. Noting the definition of $r_\xi(i) = \chi(f_\xi, g_i)$, we get $\forall i \geq n\ \forall m \geq r(i)\ f_\xi(m) < g_i(m)$. This implies that $f_\xi <_{r(n)} g$. □

Now we can prove Theorem 4.4. Suppose for a contradiction that $\omega_2 < \mathfrak{b}$. So we can define an increasing sequence $\langle f_\xi \mid \xi < \omega_2 \rangle$ of length ω_2, and we can even find a bound g_0 (in the eventual bounding relation $<^*$) to this sequence. Define by induction on $i < \mathfrak{c}$, function $g_i \in \omega^\omega$, beginning with g_0, so that g_i is an upper bound of the sequence $\langle f_\xi \mid \xi < \omega_2 \rangle$ and $g_j <^* g_i$ whenever $i < j$ as follows. At successor stages when g_i is constructed, let $g_{i+1} = g_i^-$. At limit stage we choose g_δ if possible. For some $\gamma \leq \mathfrak{c}$ the construction stops and g_γ cannot be found (so we have a gap). Clearly, γ is a limit ordinal. Since $\omega_2 < \mathfrak{b}$, Lemma 4.5 implies that $\mathrm{cf}(\gamma) > \omega$. Consider the ideals I_{g_i} for $i < \gamma$. We noticed (Lemma 4.2) that I_{g_i} increases with $i < \gamma$. Define $I = \bigcup_{i<\gamma} I_{g_i}$. Then I is again a P-ideal over ω_2 (because if is a union of a chain of P-ideals of length that has an uncountable cofinality).

By the PID assumption there are two possibilities. The first says that there is an uncountable set H inside I. We assume that H has cardinality $\aleph_1$ and pick some $\xi_0 < \omega_2$ above H. So $f_{\xi_0} <^* g_i$ for all i's, and hence $I \subseteq I_{f_{\xi_0}}$ and H is inside $I_{f_{\xi_0}}$ which we know is impossible.

The second PID possibility implies the existence of some $H \subseteq \omega_2$ that is cofinal in ω_2 and outside of I. Then H is outside of every I_g for $g = g_i$, $i < \gamma$. Thus (by Lemma 4.3) g_H fills the gap, which contradicts the definition of γ. □

5. Viale's proof of SCH

We describe here Viale's celebrated result that uses the PID in order to derive a consequence of the proper forcing axiom.

Theorem 5.1 (Viale [5]) *The Proper forcing axiom implies the singular cardinals hypothesis.*

The Singular Cardinal Hypothesis (SCH) is that if λ is a singular cardinal and $2^{\mathrm{cf}\lambda} < \lambda$ then $\lambda^{\mathrm{cf}\lambda} = \lambda^+$. Silver's theorem (1974) says that if κ is a singular cardinal with uncountable cofinality and if $\delta^{\mathrm{cf}\delta} = \delta^+$ for a stationary set of cardinals $\delta < \kappa$, then $\kappa^{\mathrm{cf}\kappa} = \kappa^+$. Using Silver's theorem, the SCH follows from the the following statement:

$$\text{for all } \kappa \geq 2^{\aleph_0},\ \text{if } \mathrm{cf}(\kappa) = \omega,\ \textit{then } \kappa^{\aleph_0} = \kappa^+. \tag{5}$$

We shall prove that (5) is a consequence of the PID by induction on λ.

For a singular cardinal κ with $\mathrm{cf}\kappa = \omega$ we define that $\{K_n^\alpha \mid \alpha < \kappa^+ \;\&\; n \in \omega\}$ is a *covering system* if for every $\alpha < \kappa^+$, $\alpha = \bigcup_n K_n^\alpha$, $K_n^\alpha \subseteq K_{n+1}^\alpha$, and $|K_n^\alpha| < \kappa$. The system is said to be *upward coherent* iff for $\alpha < \beta < \kappa^+$

$$\forall n \exists m\ K_n^\alpha \subseteq K_m^\beta.$$

An upward coherent system is weakly downward coherent if for every countable $X \subset \kappa^+$ there exists γ_X so that for every α, β with $\gamma_X \le \alpha < \beta$ for all n there exists m such that

$$X \cap K_n^\beta \subseteq K_m^\alpha.$$

Observe that, equivalently, weakly downward coherence is the statement that for every countable $X \subset \kappa^+$ there exists $\gamma = \gamma_X$ (can be taken above X) so that for all $\beta > \gamma$ for all n there exists m such that

$$X \cap K_n^\beta \subseteq K_m^\gamma.$$

(To see the equivalence, use the fact that the system is upward coherent.)

Assume $\mathrm{cf}(\kappa) = \omega$ and let $\{\kappa_n \mid n \in \omega\}$ be regular cardinals increasing towards κ. We shall define an upward coherent system and prove that any such system is weakly downward coherent (assuming $\kappa > 2^{\aleph_0}$). We shall define for every $\alpha < \kappa^+$ $K_n^\alpha \subseteq K_{n+1}^\alpha$ for $n \in \omega$ such that:

(1) $\alpha = \bigcup_n K_n^\alpha$.

(2) K_n^α has cardinality $< \kappa$ and in fact $\le \kappa_n$.

(3) Upward coherence: if $\alpha < \beta$ for every n there is m so that $K_n^\alpha \subseteq K_m^\beta$.

Define by induction on $\kappa \le \alpha < \kappa^+$ the sets K_n^α for all $n \in \omega$. For κ itself we can take $K_n^\kappa = \kappa_n$. At successors, let $K_n^{\alpha+1} = K_n^\alpha \cup \{\alpha\}$. For limit δ pick a cofinal set $C \subset \delta$ of cardinality $< \kappa$, say of cardinality $\le \kappa_{n_0}$. Define $K_n^\delta = \bigcup_{\alpha \in C} K_n^\alpha$ for $n > n_0$.

It is easy to check that the system built is upward coherent.

Lemma 5.2 *If $\kappa \ge 2^{\aleph_0}$ and $\mathrm{cf}\kappa = \omega$ then there is a covering system for κ^+ that is upward coherent and weakly downward coherent.*

Proof. First we get an upward coherent system and then check that it is automatically weakly downward coherent. Given a countable $X \subset \kappa^+$ consider

$$K^\alpha(X) = \langle K_n^\alpha \cap X \mid n < \omega \rangle.$$

Fixing X, there are $2^{\aleph_0}$ possibilities for $K^\alpha(X)$. So as $\kappa \ge 2^{\aleph_0}$ there is a fixed value $K(X)$ so that $K(X) = K^\alpha(X)$ for an unbounded set of $\alpha < \kappa^+$. Let $\gamma = \gamma_X$ be any of these α's. Given any $\beta > \gamma$, and K_n^β, pick $\alpha > \beta$ so that $K(X) = K^\alpha(X)$. Then by upward coherence $K_n^\beta \subseteq K_m^\alpha$ for some m, and hence

$$X \cap K_n^\beta \subseteq K_m^\alpha \cap X = K_m^\gamma \cap X.$$

This proves the lemma. □

We prove that (5) is a consequence of the PID by induction on κ. Since $\mathrm{cf}(\lambda) = \omega$, there is a covering system K_n^α for κ^+ that is weakly downward coherent and such that $|K_n^\alpha| < \kappa$ for all α and n. We have by the inductive assumption that $|K_n^\alpha|^{\aleph_0} < \kappa$.

Define an ideal I on $[\kappa^+]^{\le\aleph_0}$:

$$A \in I \text{ iff } A \cap K_n^\alpha \text{ is finite for all } \alpha, n.$$

Claim: For $A \in [\kappa^+]^{\le\aleph_0}$, if $A \subset \gamma$, then $A \in I$ iff for all n $A \cap K_n^\gamma$ is finite. Proof of claim. Suppose that $A \subset \gamma$ and $A \cap K_n^\gamma$ is finite for all n. If $\alpha < \gamma$ then $A \cap K_k^\alpha$ is finite by the upward coherence. If $\alpha > \gamma$, then the weak downward coherence gives the result.

From this we can conclude that I is a P-ideal. If each $A_n \in I$, form $B = \bigcup_n A_n$ and pick $\gamma > \sup B$. Then define $B^* = \bigcup_n (A_n \setminus K_n^\gamma)$. We have $B^* \in I$.

It also follows that if X is countable and "outside" of I (no infinite subset of X is in I) then for $\gamma \geq supX$ we have that $X \subset^* K_k^\gamma$ for some k. (Otherwise, there is an infinite sequence $x_i \in X$ so that $x_i \notin K_i^\gamma$, and then $\{x_i \mid i < \omega\}$ is in I.)

By the P-ideal dichotomy, there are two options.

(1) The first is that there is some uncountable $X \subset \kappa^+$ that is "inside" I. This implies that $X \cap K_n^\alpha$ is finite for all indices. This is impossible. Let α be such that $X \subset \alpha$. Then there is some n so that $X \cap K_n^\alpha$ is uncountable and hence surely infinite.

(2) $\kappa^+ = \bigcup_n A_n$ where each A_n is "outside" of I. There exists n so that $A = A_n$ has cardinality κ^+. If we prove that $|A|^{\aleph_0} = \kappa^+$ then we are done. Since A is "outside" of I, every countable subset of $X \subset A$ has some K_n^γ so that $X \subset^* K_n^\gamma$. (If a countable set $X \subset \gamma$ is not almost included in any K_n^γ then an infinite subset of X is in I.) But $|K_n^\gamma| < \kappa$ and hence $|K_n^\gamma|^{\aleph_0} < \kappa$ by the inductive assumption. Thus the number of countable subsets of A is κ^+.

6. Appendix

Since we wish these lectures to be accessible to a wider readership, we shall define here the Proper Forcing Axiom, and explain some standard arguments concerning elementary substructures that are used here. For any cardinal κ, $H(\kappa)$ denotes the collection of all sets of transitive closure with cardinality smaller than κ. We also denote with $H(\kappa)$ the structure with universe $H(\kappa)$, the $\in$ relation on that universe, and a well-ordering $<$ of that universe. The role of this well-ordering is to ensure that objets used in the proof are well-defined (when we say in a proof "let x be such that..." this description can be understood as "let x be the first in the $<$-well-ordering such that...".

We say that M is an elementary substructure of $H(\kappa)$ (and we write $M \prec H(\kappa)$) when for every sentence φ (in the language with the membership relation $\in$ and the well-ordering relation $<$) with parameters from M, $M \models \varphi$ iff $H(\kappa) \models \varphi$. We say that M is countable when its universe $|M|$ is countable. For every $X \in H(\kappa)$ there is a countable $M \prec H(\kappa)$ with $X \in M$ (by the Lowenheim–Skolem theorem).

A fact that is often used when $M \prec H(\kappa)$ is a countable elementary substructure is that if $X \in M$ is countable (in M, in $H(\kappa)$, or in the universe V–it's all the same) then $X \subset M$ (by which it is meant that $X \subset |M|$). The reason is that there is in M (by elementaricity) a function $f : \omega \to X$ that is onto X, and as $\omega \subset M$ we obtain $X \subset M$.

In particular, every countable ordinal in M is a subset of M. So $M \cap \omega_1$ is an initial segment of the ordinals–and hence an ordinal itself.

Another simple observation is that if $M \prec H(\kappa)$ is countable, then $M \in H(\kappa)$, and hence there is a countable $N \prec H(\kappa)$ with $M \in N$.

In proving properness we used without comment the following fact. If $M \prec H(\kappa)$, then $M \cap H(\aleph_2) \prec H(\aleph_2)$. (Since $\aleph_2$ is definable in $H(\kappa)$ as the second uncountable cardinal, $H(\aleph_2) \in M$ and $M \cap H(\aleph_2) = H(\aleph_2)^M$. So if $M \cap H(\aleph_2) \models \varphi$, then it holds in M that $H(\aleph_2) \models \varphi$, and hence this is so in $H(\kappa)$, and thus indeed $H(\aleph_2) \models \varphi$.)

Given $M \prec H(\kappa)$ with $P \in M$, we say that a condition $p \in P \cap M$ is (M, P)-generic if for every dense set D of P that is member of M, every extension of p is compatible with some member of $D \cap M$. (That is, for every $q \leq p$ there exists $d \in D \cap M$ so that $q' \leq d$ for some $q' \leq q$.)

A poset P is said to be *proper* if for every cardinal $\kappa > 2^{|P|}$, for every countable $M \prec H(\kappa)$ with $P \in M$, every $p_0 \in P \cap M$ has an extension $p \leq p_0$ that is (M, P)-generic.

The Proper Forcing Axiom is the statement that if P is any proper poset and $\{D_\alpha \mid \alpha < \omega_1\}$ are dense sets, then there is a filter in P that intersects every D_α.

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