PANM 12

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The problem with bilateral contact and nonmonotone friction

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THE PROBLEM WITH BILATERAL CONTACT AND NONMONOTONE FRICTION

Zuzana Morávková

1. Problem formulation

We study the generalized plain strain problem with standard boundary conditions and bilateral contact and nonmonotone friction conditions.

The domain $\Omega = 100\,mm \times 10\,mm$ and parts Γ_U, Γ_P, Γ_C of the boundary $\partial\Omega$ are depicted on Figure 1.

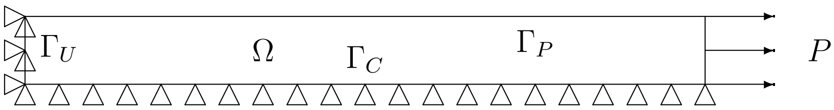

Fig. 1.

The body is made of a linear isotropic material obeying the generalized plane strain model characterized by the modulus of elasticity $E = 2.1\,10^5 N/mm^2$, Poisson's ratio $\sigma = 0.3$, and the element thickness $t = 5\,mm$. The structure is fixed along Γ_U, i.e. the zero displacements in both directions are prescribed:

$$u_i = 0 \quad \text{on } \Gamma_U,\ i = 1,2\,. \tag{1}$$

On the right side of the body the perpendicular surface traction $T = (P, 0)$, where $P = const$, $P \geq 0$ a.e. on Γ_P is applied (see Figure 1):

$$T_1 = P \quad \text{on } \Gamma_P\,. \tag{2}$$

Now we define the nonmonotone friction of various types on Γ_C (see Figure 2). Moreover we prescribe the bilateral condition:

$$-T_1(x) \in \hat{b}(u_1(x)) \quad x \in \Gamma_C\,, \qquad u_2 = 0 \quad \text{on } \Gamma_C\,. \tag{3}$$

The boundary conditions (1)–(3) are completed with the system of equilibrium equations:

$$\frac{\partial \tau_{ij}(u)}{\partial x_j} = 0 \quad \text{in } \Omega, \qquad i = 1,2\,. \tag{4}$$

The volume forces are equal to zero.

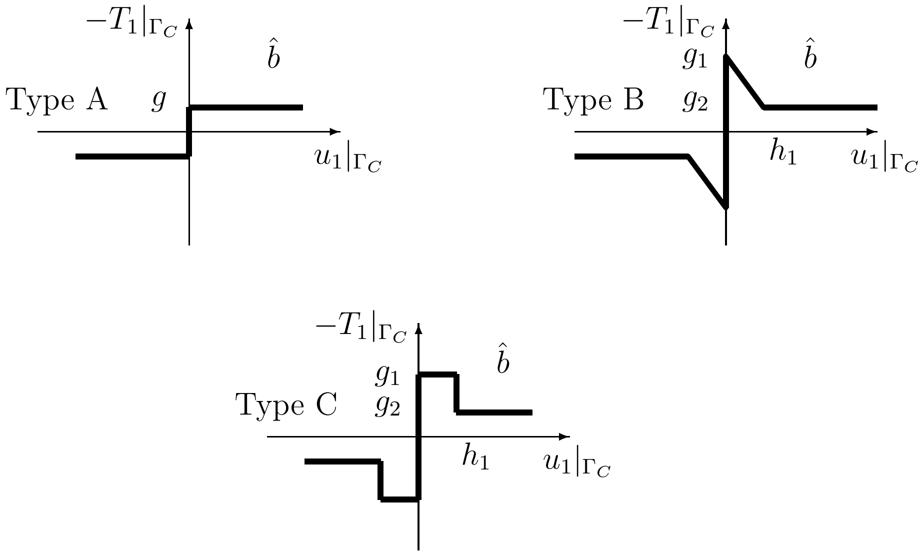

Fig. 2.

The stress tensor $\{\tau_{ij}\}_{i,j=1}^{2}$ is related to the linearized strain tensor $\{\varepsilon_{ij}\}_{i,j=1}^{2}$ by means of the following linear generalized Hooke's law:

$$\tau_{ij}(u) = \frac{E\sigma}{1-\sigma^2}\delta_{ij}\vartheta + \frac{E}{1+\sigma}\varepsilon_{ij}(u), \quad i,j = 1,2\,, \tag{5}$$

where $\vartheta = \varepsilon_{ii}$ is the trace of $\{\varepsilon_{ij}\}_{i,j=1}^{2}$ and δ_{ij} is the Kronecker symbol.

By a *classical solution* of the problem with bilateral contact and nonmonotone friction we call any displacement field $u = (u_1, u_2)$ satisfying the boundary conditions (1), (2), (3) and the equations (4) with the linear Hooke's law (5).

In order to give the weak formulation of the previous problem we introduce the following notations:

$$\begin{aligned} V &= \{v \in (H^1(\Omega))^2 |\ v = 0 \text{ on } \Gamma_U, \quad v_2 = 0 \text{ on } \Gamma_C\} \\ a(u,v) &= \int_\Omega \tau_{ij}(u)\varepsilon_{ij}(v)\,dx\,, \\ L(v) &= \int_{\Gamma_P} Pv\,ds\,. \end{aligned}$$

The *weak formulation* of the problem with *bilateral contact* and *nomonotone friction* is given by the following hemivariational equality (see [1]):

$$\left\{ \begin{array}{l} \textit{Find } (u,\Xi) \in V \times L^2(\Gamma_C) \textit{ such that} \\ a(u,v) + \int_{\Gamma_C} \Xi v_1 dx_1 = L(v) \quad \forall v \in V \\ \Xi(x) \in \hat{b}(u_1(x)) \quad \textit{for a.a. } x \in \Gamma_C\,. \end{array} \right. \tag{6}$$

The problem has at least one solution $(u,\Xi) \in V \times L^2(\Gamma_C)$, see [1].

2. Discretization

We briefly describe the discretization of (6). Let $\mathcal{D}_h$, $h \to 0+$ be a system of regular triangulations of $\overline{\Omega}$ (see Figure 3).

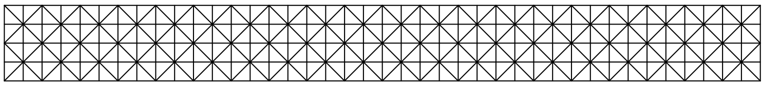

Fig. 3.

Let us define the space of all continuous piecewise linear vector functions on $\mathcal{D}_h$: $V_h = \{v_h = (v_{h1}, v_{h2})) \in (C(\overline{\Omega}))^2 |\, v_{h|T} \in (P_1(T))^2\, \forall T \in \mathcal{D}_h;\ v_h = 0 \text{ on } \Gamma_U,\ \ v_{h2} = 0 \text{ on } \Gamma_C\}$.

Now we construct the space Y_h. By $\{x_h^i\}_{i=1}^m$ we denote the set of all nodes of triangulation $\mathcal{D}_h$ on the part of boundary $\overline{\Gamma}_C \backslash \overline{\Gamma}_U$. Let $x_h^{i+1/2}$ be the midpoint of the interval $[x_h^i, x_h^{i+1}]$, $i = 0, \ldots, m-1$. The partion $\mathcal{T}_h$ of $\overline{\Gamma}_C$ defining the space Y_h consists of all segments S_i joining the midpoints $x_h^{i-1/2}$, $x_h^{i+1/2}$, $i = 2, \ldots, m-1$ with the modifications concerning S_1 and S_m. Let $S_1 = [x_h^0, x_h^{3/2}]$, $S_m = [x_h^{m-1/2}, x_h^m]$ (see Figure 4).

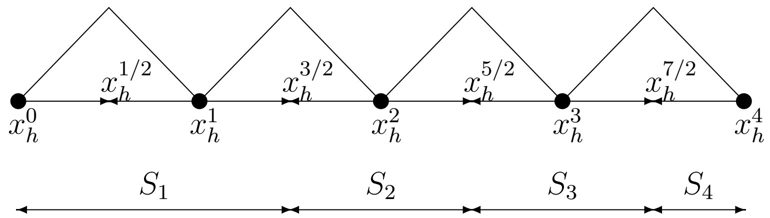

Fig. 4.

On any such partition $\mathcal{T}_h$ we shall define the space Y_h of all piecewise constant functions with values at $\{x_h^i\}_{i=1}^m$ as the degrees of freedom. The space W_h consists of all continuous piecewise linear scalar functions over the partition defined by the nodes $\{x_h^i\}_{i=0}^m$ and vanishing at the initial node x_h^0. Due to the definition of Y_h we also see that $\dim W_h = \dim Y_h$.

We define the mapping $P_h : \ W_h \to Y_h$ by $P_h(w_h) = \sum_{i=1}^m w_h(x_h^i)\chi_{S_i}(x_1)$, $w_h \in W_h$, where χ_{S_i} is the characteristic function of the interior of S_i. This mapping associates with a function $w_h \in W_h$ its piecewise constant Lagrange interpolate on $\mathcal{T}_h$ (see Figure 5).

We are able to define the *discrete hemivariational equality* approximating (6):

$$\begin{cases} \textit{Find } (u_h, \Xi_h) \in V_h \times Y_h \textit{ such that} \\ a(u_h, v_h) + \int_{\Gamma_C} \Xi_h P_h v_{h1} dx_1 = L(v_h) \qquad \forall v_h = (v_{h1}, v_{h2}) \in V_h \\ \Xi_h(x_h^i) \in \hat{b}(P_h(u_{h1})(x_h^i)) \quad \forall i = 1, \ldots, m\,. \end{cases} \tag{7}$$

The existence of the solutions and convergence results are described in [1].

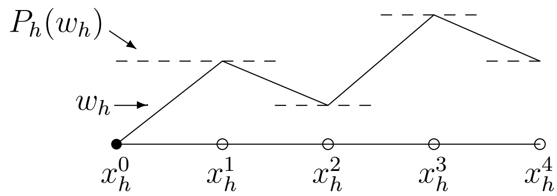

Fig. 5.

Let us introduce the algebraic form of the problem (7). We suppose that the points $\{x_h^i\}_{i=0}^m$ are equidistant on $\overline{\Gamma}_C$. We denote $\dim V_h = n$, $\dim Y_h = m$.

We define $\Xi_i := c_i \Xi_i$, where $c_1 = 3/2h$, $c_2 = \cdots = c_{m-1} = h$, $c_m = h/2$. The problem (7) can be rewritten in the following form:

$$\left\{\begin{array}{l} \textit{Find } (\boldsymbol{u}, \Xi) \in \mathbb{R}^n \times \mathbb{R}^m \textit{ such that} \\ (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^n} + (\Xi, \boldsymbol{\Lambda}(\boldsymbol{v}))_{\mathbb{R}^m} = (\boldsymbol{f}, \boldsymbol{v})_{\mathbb{R}^n} \quad \forall \boldsymbol{v} \in \mathbb{R}^n \\ \Xi_i \in c_i \hat{b}((\boldsymbol{\Lambda}\boldsymbol{u})_i) \quad \forall i = 1, \ldots, m\,, \end{array}\right. \tag{8}$$

where $(\boldsymbol{\Lambda}\boldsymbol{v})_i :=$ x_1-component of vector $\boldsymbol{v}$ in points x_h^i, $i = 1, \ldots, m$.

Since the bilinear form a is symmetric, we can construct discrete superpotential $\mathcal{L}$, which has a quadratic part and lipschitz continuous perturbation Ψ defined by the rectangular formula $\Psi(\boldsymbol{v}) = \sum_i c_i \Phi((\boldsymbol{\Lambda}\boldsymbol{v})_i)$, where Φ is a primitive function to b. (The function b was derived from $\hat{b}$ by leaving the vertical parts in the graph out.)

The discrete superpotential $\mathcal{L}$ corresponding to the algebraic hemivariational equality has the form:

$$\mathcal{L}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v})_{\mathbb{R}^n} - (\boldsymbol{f}, \boldsymbol{v})_{\mathbb{R}^n} + \Psi(\boldsymbol{v}), \;\; \boldsymbol{v} \in \mathbb{R}^n\,.$$

Instead of the problem (8) we shall consider the following *substationary type problem*:

$$\left\{\begin{array}{l} \textit{Find } \boldsymbol{u} \in \mathbb{R}^n \textit{ such that} \\ \boldsymbol{0} \in \overline{\partial}\mathcal{L}(\boldsymbol{u})\,, \end{array}\right. \tag{9}$$

where $\overline{\partial}$ denotes the generalized gradient in the sense of Clark.

If mapping P_h maps W_h *onto* Y_h, then the problems (8) and (9) are equivalent under the assumption of existence one-sides limits $b(\xi\pm)$ for all $\xi \in \mathbb{R}$. The problem (9) can be solved by using nonsmooth bundle type minimization methods (see [2]). Knowing $\boldsymbol{u}$ in (9) one can recover also the vector $\boldsymbol{\Xi}$ from relation $\boldsymbol{\Lambda}^T\boldsymbol{\Xi} = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}$, such that the pair $(\boldsymbol{u}, \boldsymbol{\Xi})$ solves (8).

3. The examples

The geometrical and material characteristics are described in Section 1., traction $P = 0.05\, N/mm^2$ and the parameter values in diagrams in Figure 2 are $h_1 = 4 \cdot 10^{-6}$, $g_1 = 5$, $g_2 = 4$.

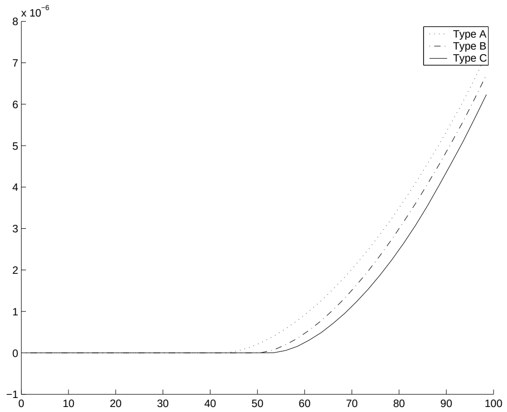

Fig. 6.

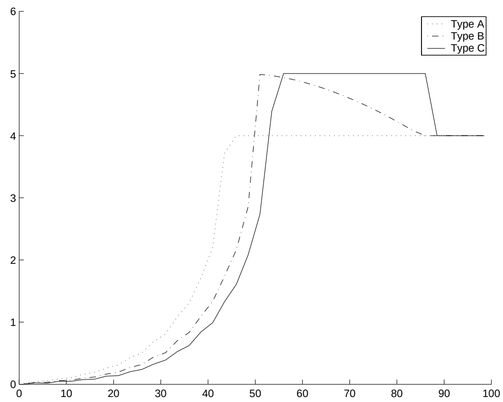

Fig. 7.

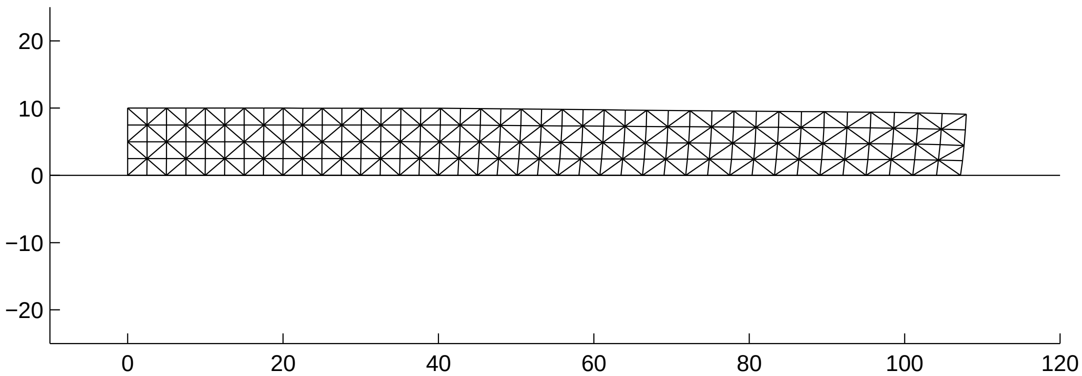

Fig. 8.

Figures 6 and 7 illustrate the graph of the tangential components of the displacement and stress vector along the contact part Γ_C, respectively. Finally the total deformation of Ω enlarged $10^6\times$ is depicted in Figure 8.

References

[1] J.Haslinger, M.Miettinnen, P.D.Panagiotopoulos: *Finite element method for hemivariational inequalities - theory, methods and applications*. Nonconvex Optimization and its Applications **35**, Kluwer Academic Publishers, Dordrecht, 1999.

[2] L. Lukšan, J. Vlček: *A bundle–Newton method for nonsmooth unconstrained minimization*. Math. Prog. **83**, 1998, 373–391.