Libor Čermák

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LINEAR STABILITY OF EULER EQUATIONS IN CYLINDRICAL DOMAIN*

Libor Čermák

Abstract

The linear stability problem of inviscid incompressible steady flow between two concentric cylinders is investigated. Linearizing the transient behavior around a steady state solution leads to an eigenvalue problem for linearized Euler equations. The discrete eigenvalue problem is obtained by the spectral element method. The algorithm is implemented in MATLAB. The developed program serves as a simple tool for numerical experimenting. It enables to state rough dependency of the stability on various input velocity profiles.

1. Flow equations in the rotating cylindrical coordinate system

The inviscid incompressible flow in the cylindrical coordinate system (r, φ, z) rotating about the z-axis with the angular velocity Ω_0 is described by Euler equations

$$\begin{aligned}
&\frac{\mathrm{d}w_r}{\mathrm{d}t} - \frac{w_\varphi^2}{r} - 2\Omega_0 w_\varphi - \Omega_0^2 r + \frac{1}{\varrho}\frac{\partial p}{\partial r} = 0\,,\\
&\frac{\mathrm{d}w_\varphi}{\mathrm{d}t} + \frac{w_r w_\varphi}{r} + 2\Omega_0 w_r + \frac{1}{\varrho r}\frac{\partial p}{\partial \varphi} = 0\,,\\
&\frac{\mathrm{d}w_z}{\mathrm{d}t} + \frac{1}{\varrho}\frac{\partial p}{\partial z} = 0\,.
\end{aligned} \tag{1}$$

Here, w_r, w_φ, and w_z are radial, circumferential and axial velocities, p is the pressure,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + w_r\frac{\partial}{\partial r} + \frac{w_\varphi}{r}\frac{\partial}{\partial \varphi} + w_z\frac{\partial}{\partial z} \tag{2}$$

is the material derivative and ϱ is the density. The continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r w_r) + \frac{1}{r}\frac{\partial w_\varphi}{\partial \varphi} + \frac{\partial w_z}{\partial z} = 0 \tag{3}$$

must be fulfilled as well. Equations (1)–(3) are presented, for example, in [1]. The problem is solved in the domain Q between two coaxial cylinders,

$$Q = \{(r, \varphi, z)\,|\, 0 < R_1 \le r \le R_2, 0 \le \varphi < 2\pi, 0 \le z \le L\}\,, \tag{4}$$

where $S_1 = Q \cap \{z = 0\}$ and $S_2 = Q \cap \{z = L\}$ are the pipe inlet and outlet, respectively, $\Gamma_1 = Q \cap \{r = R_1\}$ is the liquid-gas interface and $\Gamma_2 = Q \cap \{r = R_2\}$ is the pipe wall.

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2. Linear stability

Let us suppose that the steady base flow is axially symmetric, described by functions $w_{0r}(r,z)$, $w_{0\varphi}(r,z)$, $w_{0z}(r,z)$, and $p_0(r,z)$ and by corresponding boundary conditions. To investigate the stability of the base flow to disturbances, equations that govern the evolution of these perturbations are required. To this end, the base flow is perturbed by disturbance velocities and pressure, i.e.

$$(w_r, w_\varphi, w_z, p) = (w_{0r}, w_{0\varphi}, w_{0z}, p_0) + \varepsilon(v_r, v_\varphi, v_z, \sigma)\,, \tag{5}$$

and we examine, whether $(w_r, w_\varphi, w_z, p) \to (w_{0r}, w_{0\varphi}, w_{0z}, p_0)$ for $t \to \infty$. If we substitute from (5) into (1), (3), use the fact that the stationary flow functions w_{0r}, $w_{0\varphi}$, w_{0z}, p_0 satisfy those equations, and if we neglect terms containing ε^2, we obtain linearized Euler equations

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + w_{0r}\frac{\partial v_r}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_r}{\partial \varphi} + w_{0z}\frac{\partial v_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}v_r + \frac{\partial w_{0r}}{\partial z}v_z - \\
&\qquad - \frac{2}{r}w_{0\varphi}v_\varphi - 2\Omega_0 v_\varphi + \frac{1}{\varrho}\frac{\partial \sigma}{\partial r} = 0\,, \\
&\frac{\partial v_\varphi}{\partial t} + w_{0r}\frac{\partial v_\varphi}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_\varphi}{\partial \varphi} + w_{0z}\frac{\partial v_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}v_r + \frac{\partial w_{0\varphi}}{\partial z}v_z + \\
&\qquad + \frac{w_{0r}}{r}v_\varphi + \frac{w_{0\varphi}}{r}v_r + 2\Omega_0 v_r + \frac{1}{\varrho r}\frac{\partial \sigma}{\partial \varphi} = 0\,, \\
&\frac{\partial v_z}{\partial t} + w_{0r}\frac{\partial v_z}{\partial r} + \frac{w_{0\varphi}}{r}\frac{\partial v_z}{\partial \varphi} + w_{0z}\frac{\partial v_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}v_r + \frac{\partial w_{0z}}{\partial z}v_z + \frac{1}{\varrho}\frac{\partial \sigma}{\partial z} = 0,
\end{aligned} \tag{6}$$

and the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0\,. \tag{7}$$

Boundary conditions are

$$v_r = v_\varphi = v_z = 0 \quad \text{on } S_1, \qquad \sigma = 0 \quad \text{on } S_2, \qquad v_r = 0 \quad \text{on } \Gamma_2. \tag{8}$$

We consider two types of boundary conditions on Γ_1:

(a) if the surface tension effect is not taken into account, zero pressure is described,

$$\sigma = 0 \qquad \text{on } \Gamma_1\,; \tag{9}$$

(b) if the surface tension effect is taken into account, we proceed in accordance with the ideas derived in [7] and [8]: we introduce the radial displacement $\Delta(\varphi, z, t)$ of the boundary Γ_1 and demand fulfillment of the impermeability equation

$$v_r = \frac{\partial \Delta}{\partial t} + w_{0z}\frac{\partial \Delta}{\partial z} \qquad \text{on } \Gamma_1\,, \tag{10}$$

the Young-Laplace equation

$$\sigma = \sigma_p \left(\frac{\partial^2 \Delta}{\partial z^2} + \frac{1}{R_1^2} \frac{\partial^2 \Delta}{\partial \varphi^2} \right) \qquad \text{on } \Gamma_1 \,, \tag{11}$$

where σ_p is the surface tension coefficient, and the initial condition

$$\Delta = 0 \qquad \text{on} \quad \Gamma_1 \cap S_1 \,. \tag{12}$$

The stability problem consists in verifying whether

$$(v_r, v_\varphi, v_z, \sigma, \Delta) \to (0,0,0,0,0) \qquad \text{for } t \to \infty. \tag{13}$$

If the condition (9) is prescribed on the boundary Γ_1, we set $\Delta = 0$ in (13).

3. The eigenvalue problem

Let $I = \sqrt{-1}$ be the imaginary unit and n be a positive whole number (so-called azimuthal wave number). Using the transformation

$$(v_r, v_\varphi, v_z, \sigma, \Delta) = e^{\lambda t + In\varphi}(u_r, u_\varphi, u_z, h, \delta), \tag{14}$$

we exclude the time t as well as the coordinate φ and obtain the eigenvalue problem

$$\lambda u_r + w_{0r}\frac{\partial u_r}{\partial r} + w_{0\varphi}\frac{nI}{r}u_r + w_{0z}\frac{\partial u_r}{\partial z} + \frac{\partial w_{0r}}{\partial r}u_r + \frac{\partial w_{0r}}{\partial z}u_z - {}$$

$$- \frac{2}{r}w_{0\varphi}u_\varphi - 2\Omega_0 u_\varphi + \frac{1}{\varrho}\frac{\partial h}{\partial r} = 0 \,, \tag{15}$$

$$\lambda u_\varphi + w_{0r}\frac{\partial u_\varphi}{\partial r} + \frac{nI}{r}w_{0\varphi}u_\varphi + w_{0z}\frac{\partial u_\varphi}{\partial z} + \frac{\partial w_{0\varphi}}{\partial r}u_r + \frac{\partial w_{0\varphi}}{\partial z}u_z + {}$$

$$+ \frac{w_{0r}}{r}u_\varphi + \frac{w_{0\varphi}}{r}u_r + 2\Omega_0 u_r + \frac{nI}{\varrho r}h = 0 \,, \tag{16}$$

$$\lambda u_z + w_{0r}\frac{\partial u_z}{\partial r} + \frac{nI}{r}w_{0\varphi}u_z + w_{0z}\frac{\partial u_z}{\partial z} + \frac{\partial w_{0z}}{\partial r}u_r + \frac{\partial w_{0z}}{\partial z}u_z + \frac{1}{\varrho}\frac{\partial h}{\partial z} = 0 \,, \tag{17}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{nI}{r}u_\varphi + \frac{\partial u_z}{\partial z} = 0 \,. \tag{18}$$

Here, λ is an eigenvalue and u_r, u_φ, u_z, and h are eigenfunctions (so-called normal modes) in variables r and z defined on the rectangle

$$D = \{(r,z) \,|\, 0 < R_1 \le r \le R_2, 0 \le z \le L\} \,. \tag{19}$$

The case $n = 0$ corresponds to the rotationally symmetric flow. If we take into consideration the influence of the surface tension, we add moreover equations

$$\lambda \delta(z) + w_{0z}(R_1, z)\delta'(z) - u_r(R_1, z) = 0 \,, \tag{20}$$

$$\sigma_p \left(-\delta''(z) + \frac{n^2}{R_1^2}\delta(z) \right) + h(R_1, z) = 0 \,, \tag{21}$$

where $\delta(z)$, $z \in \langle 0, L\rangle$, is a function defined on the edge $r = R_1$ of the domain D.

Boundary conditions are

$$\begin{aligned} u_r = u_\varphi = u_z = 0 & & & S_1 \cap D\,, \\ u_r = 0 & & \text{on} \quad & \Gamma_2 \cap D\,, \\ h = 0 & & & S_2 \cap D\,. \end{aligned} \tag{22}$$

If we do not consider surface tension influences, then the additional boundary condition is

$$h = 0 \qquad \text{on} \quad \Gamma_1 \cap D\,, \tag{23}$$

whereas in case, when the surface tension is considered, we have, besides equations (20), (21), the additional condition

$$\delta = 0 \qquad \text{for} \quad z = 0\,. \tag{24}$$

The stability, expressed by the relation (5), occurs if and only if all eigenvalues of the problem (15)–(24) have negative real parts.

4. The discretization by the spectral element method

The approximate finite-dimensional eigenvalue problem is obtained by the spectral element method, see e.g. [2], [6], [4]. Let us explain in brief how the approximate eigenvalue problem can be obtained.

The rectangle D is divided into $n_r \times n_z$ concurrent rectangular elements D^{ij}, $i = 1, 2, \ldots, n_r$, $j = 1, 2, \ldots, n_z$, with side lengths $d_r = (R_2 - R_1)/n_r$ and $d_z = L/n_z$ in the discretization of the r-axis and z-axis, respectively. All quantities u_r, u_φ, u_z, h, w_{0r}, $w_{0\varphi}$ and w_{0z} are approximated by continuous piecewise polynomial functions, which are on every element D^{ij} polynomials of degree N uniquely determined by their values at nodes of the Gauss-Legendre-Lobatto (GLL) product quadrature formula of order $2N - 1$, see e.g. [6]. If the surface tension is considered, δ is similarly approximated by a continuous piecewise polynomial function, which is on every element D^{1j} polynomial of degree N uniquely determined by its values at nodes of the GLL quadrature formula.

Further, the variational formulation is derived. Let ψ_r, ψ_φ, ψ_z and ψ_h be test functions of the same type as corresponding piecewise polynomial approximations of u_r, u_φ, u_z, and h. Equations (15), (16), (17), and (18) are multiplied by ψ_r, ψ_φ, ψ_z and ψ_h and integrated over the domain D. The integral over the whole domain is expressed as a sum of integrals over individual elements D^{ij} and every from those integrals is computed by the GLL product quadrature formula. Variational forms connected with equations (20) and (21) are obtained similarly.

If we sum all equations and arrange unknown and free parameters in column vectors $\mathbf{u}$ and $\boldsymbol{\psi}$, respectively, we obtain

$$\boldsymbol{\psi}^T(\lambda\mathbf{B} - \mathbf{A})\mathbf{u} = 0\,,$$

and as the vector $\boldsymbol{\psi}$ is arbitrary, we arrive at the generalized eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{B}\mathbf{u}\,. \tag{25}$$

The matrix $\mathbf{A}$ is regular, nonhermitian, real for $n = 0$ and complex for $n > 0$. The matrix $\mathbf{B}$ is diagonal and singular: diagonal coefficients corresponding to equations (18) and (21) are equal to zero, remaining diagonal coefficients are real and positive. Infinite eigenvalues have no influence on the stability examination and therefore we ignore them.

5. Numerical experiments

Example 1. Let us consider a hypothetical flow of water as if it was a solid body movement, set

$$\begin{array}{llll} R_1 = 0.015 & [\mathrm{m}] & R_2 = 0.15 & [\mathrm{m}] \\ L = 0.5 & [\mathrm{m}] & \varrho = 10^3 & [\mathrm{kg}\cdot\mathrm{m}^{-3}] \\ \sigma_p = 0.073 & [\mathrm{N}\cdot\mathrm{m}] & n = 0,1,2,3 & \end{array} \tag{26}$$

$w_{0r} = 0$, $w_{0\varphi} = 0$, $w_{0z} = C_0 = \text{const.}$ and experiment with values of $C_0 \geq 0$, Ω_0, n_r, n_z, and N. The flow of this type is stable and numerical results have confirmed this.

Remark. Through numerous numerical tests based on the data (26) we have found that the effects of surface tension are negligible. It is a good message saying that a steady state solution can be computed using any CFD software (whereas considering surface tension influences do not belong to standard equipment of commercial CFD software, setting the pressure on the boundary is a quite common instrument).

Example 2. We consider the case when the stationary velocity in the radial and axial direction are constant, $w_{0r} = 0$, $w_{0z} = C_0 \geq 0$, and the circumferential velocity is a function of the radial variable r,

$$w_{0\varphi} = r(ae^{-r/b} - \Omega_0)\,. \tag{27}$$

Constants $a > 0$, $b > 0$ are optional parameters: $c_{0\varphi} = w_{0\varphi} + \Omega_0 r = are^{-r/b}$ approaches its maximal value for $r = b$, a influences $\max|c_{0\varphi}(r)|$. We use again the data (26).

If we set $C_0 = 0$, $n = 0$ (which means that only axisymmetric perturbations were permitted) and instead of the boundary condition (23) we demanded $u_r = 0$ for $r = R_1$, we obtained the stability just when $R_2 < 2b$, which is in coincidence with the well known Rayleigh's criterion, see e.g. [5]. Further, an increase of C_0 caused an increase of the stability (i.e. $\max \mathrm{Re}(\lambda)$ of all finite λ was decreased).

For $C_0 = 0$, $n > 0$ we did not obtain the stability for neither a nor b. If C_0 was increased, the stability turned up (for appropriate values of a and b).

Another velocity profiles for $c_{0\varphi} = w_{0\varphi} + \Omega_0 r$ were created interactively. A given set of discrete points was interpolated by means of a cubic spline and then the dependence of the stability on the velocity $w_{0\varphi}$ and parameters C_0, Ω_0, n, N, n_r, n_z was examined. The obtained results fulfilled our expectancy.

Example 3. The steady state flow velocities w_{0r}, $w_{0\varphi}$, and w_{0z} were computed by the CFD package FLUENT with the data $R_1 = 0.015$, $R_2 = 0.15$, $L = 1$, $\varrho = 10^3$ and with the boundary conditions

$$\begin{aligned}
&\text{on the inlet } z = 0: && w_{0r} = 0, \quad w_{0\varphi} = r(7.5e^{-r/0.05} - 5), \quad w_{0z} = 1,\\
&\text{on the outlet } z = L: && p_0 = p_{outlet},\\
&\text{on the interface } r = R_1: && p_0 = p_{interface},\\
&\text{on the wall } r = R_2: && w_{0r} = 0.
\end{aligned}$$

Here, p_{outlet} and $p_{interface}$ are the constant pressure invoking cavitating vortex rope and the saturated vapour pressure, respectively. The surface tension influences were not taken into account. The stability examination was performed for the following parameter values: $\Omega_0 = 5$, $n_r = 4$, $n_z = 1$, $N = 8$, and $n = 0, 1, 2$. The stability of the flow, resulting from the transient FLUENT modeling, was confirmed: all finite eigenvalues had negative real parts.

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