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NEW MODEL OF PRECESSION, VALID IN TIME INTERVAL 400 THOUSAND YEARS

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Abstract

Precession is the secular and long-periodic component of the motion of the Earth's spin axis in the celestial reference frame, approximately exhibiting a motion of about $50''$ per year around the pole of the ecliptic. The presently adopted precession model, IAU2006, approximates this motion by polynomial expansions of time that are valid, with very high accuracy, in the immediate vicinity (a few centuries) of the reference epoch J2000.0. For more distant epochs, this approximation however quickly deviates from reality. As a reaction to this problem, a new model, comprising very long-period terms fitted to a numerical integration of the motion of solar system bodies on scales of several thousand centuries, was recently published by the present author with co-authors from France and United Kingdom in Astronomy & Astrophysics. Here a shorter description of the new model, including a new assessment of its accuracy and comparisons with other models, is given.

1. Introduction

The axis of rotation of the Earth is not stable in the inertial reference frame, i.e., among the stars. Under the dominant influence of the Moon and the Sun, it exhibits a rather complicated motion, called precession-nutation. Its very long-periodic part, precession, is the slow motion of the pole of Earth's rotation around the pole of the ecliptic. The angle between the two poles (obliquity) is approximately constant, roughly equal to $23.5°$. Precession was known already to Hipparchos, since it causes the growth of ecliptical longitudes of the stars by about $50''$ per year; the axis of rotation of the Earth makes one revolution in about 26 thousand years. This motion is however not so simple: the pole of the ecliptic itself is not stable with respect to the stars – it exhibits so called precession of the ecliptic (formerly called planetary precession). It is dominantly caused by the attractive forces of all bodies of the solar system on the motion of the Earth around the barycenter of the solar system. The axis of rotation of the Earth exhibits a motion around the moving pole of ecliptic

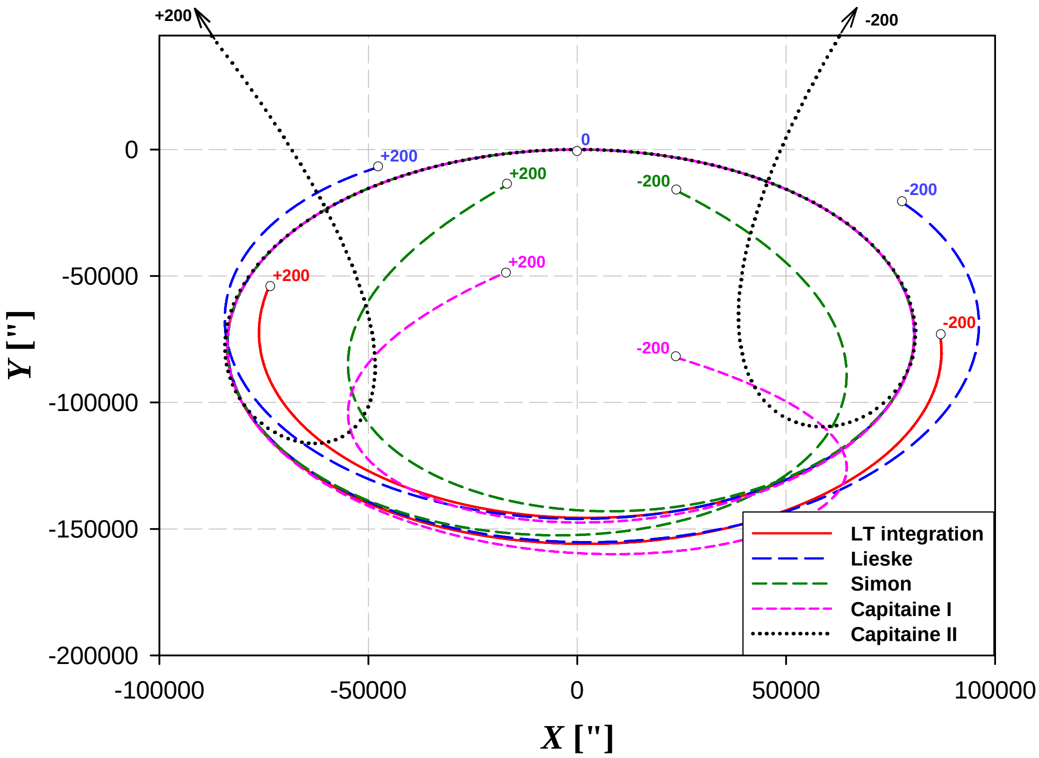

Figure 1: Different models of precession in the interval ±200cy around J2000.0.

under the torques exerted by the Moon, Sun, and planets on the rotating oblate Earth, called precession of the equator (formerly luni-solar precession), but neither obliquity, nor the rate of precession are strictly constant.

All precession models used so far are expressed in terms of polynomial development of time, no matter which of the many precession parameters (see below) are used. Model IAU2006 [2] is very accurate, but usable only for a limited time interval (several centuries around the epoch J2000); its errors rapidly increase with longer time spans. In reality, precession represents a complicated, very long-periodic process, with periods of hundreds of centuries. This can be seen in numerically integrated equations of motion of the Earth in the solar system and its rotation [11], [12]. Fig. 1 (here reproduced from paper [11]) displays the motion of the axis of rotation of the Earth during about 1.5 precession cycles, as given by long-term numerical integration (LT integration) and different analytical models – Lieske et al. [7], Simon et al. [8], and two models by Capitaine et al. [2] (computed from the expansions of precession angles ζ_A, θ_A and of direction cosines X_A, Y_A, respectively). The position of the axis of rotation at the basic epoch J2000.0 is the point $X = Y = 0$, pole of the ecliptic is approximately in the center of the figure. The models are not graphically distinguishable in the interval ±50cy around J2000, but they start to differ significantly outside the interval ±100cy.

We assume that precession covers only periods longer than 100 centuries; shorter ones are included in the nutation. The goal is to find relatively simple expressions of different precession parameters, with accuracy comparable to the IAU2006 model near the epoch J2000.0, and lower accuracy outside the interval ±1000 years (up to several minutes of arc at the extreme epochs ±200 thousand years). The paper describing the new model in detail has recently been published [13]. Below is the concise description of the model, followed by an assessment of its accuracy and comparison with other models.

2. Numerical integration, long-term expressions of precession parameters

We used the numerically integrated values of the following four parameters

- the precession of the ecliptic $P_A = \sin\pi_A \sin\Pi_A, Q_A = \sin\pi_A \cos\Pi_A$, calculated with the Mercury 6 package by Chambers [3], and
- the general precession/obliquity p_A, ε_A, provided by Laskar et al. [5]

to calculate time series for the other precession parameters in the interval ±200 thousand years from J2000.0, with 100-year steps.

To estimate the precision of the numerical integrations above, we tested them against the values obtained independently:

- Precession of the ecliptic P_A, Q_A (in which relativistic effects were neglected) was compared with the values $p = \sin i/2 \sin\Omega$, $q = \sin i/2 \cos\Omega$ (where $i = \pi_A$ is the inclination and $\Omega = \Pi_A$ longitude of the ascending node of the Earth's orbit with respect to the plane of ecliptic for J2000.0), obtained by Laskar et al. [5] by a different method with slightly different initial values, relativistic effects included. Obvious relations $P_A = 2p\sqrt{1-p^2-q^2}, Q_A = 2q\sqrt{1-p^2-q^2}$ were used, and comparison showed that the differences are only a few milliarcseconds near the epoch J2000.0 and do not exceed 20 arcseconds at the extreme epochs. The neglected perturbations by asteroids have recently been shown by Aljabaae and Souchay [1] to be very small - peak to peak quasi-periodic effects in Earth's inclination are smaller than $0.05''$, the periods are typically shorter than 100 years.
- Similarly, the comparison between different numerical integrations of the obliquity ε_A by Laskar et al. ([5], [6]) demonstrates that the differences do nor exceed the level of several arcseconds at the extreme epochs.

Thus we concluded that the precision of the numerical integration, including both numerical errors and imperfections of the model used, is sufficient for our purpose.

The central part (±1000 years from the epoch J2000.0) was then replaced by IAU2006 values to make the new model consistent with the model accepted by the IAU. From the values of the precession parameters P_A, Q_A, p_A and ϵ_A, different precession parameters were calculated in the interval ±200 millennia from J2000.0,

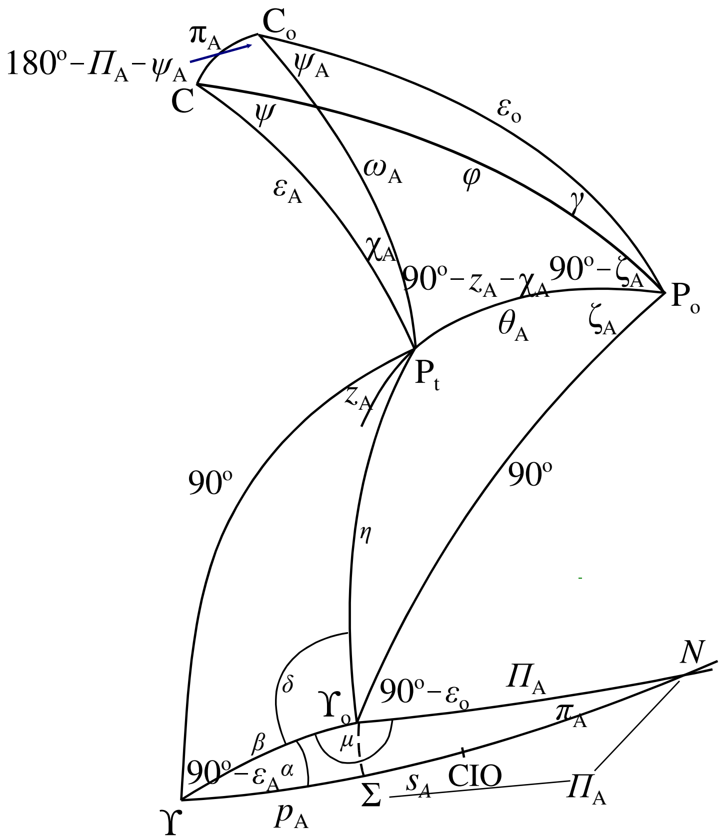

Figure 2: Precession parameters.

solving several spherical triangles depicted in Fig. 2. $C_\circ$ and C denote the positions of the pole of ecliptic at the epochs J2000.0 and T, respectively, $P_\circ$, P are the poles of rotation of the Earth and $\Upsilon_\circ$, Υ vernal points at the same epochs, CIO stands for Celestial Intermediate Origin.

We obtained first the auxiliary angles α, β, μ from the spherical triangle $\Upsilon\Upsilon_\circ$N:

$$\begin{aligned}
\cos\beta &= \cos\Pi_A\cos(\Pi_A + p_A) + \sin\Pi_A\sin(\Pi_A + p_A)\cos\pi_A \\
\sin\beta\sin\alpha &= \sin\Pi_A\sin\pi_A \\
\sin\beta\cos\alpha &= \cos\Pi_A\sin(\Pi_A + p_A) - \sin\Pi_A\cos(\Pi_A + p_A)\cos\pi_A \\
\sin\beta\sin\mu &= \sin(\Pi_A + p_A)\sin\pi_A \\
\sin\beta\cos\mu &= \sin\Pi_A\cos(\Pi_A + p_A) - \cos\Pi_A\sin(\Pi_A + p_A)\cos\pi_A,
\end{aligned} \tag{1}$$

then the angles η, δ by solving the triangle $\Upsilon\Upsilon_\circ P_t$:

$$\begin{aligned}
\cos\eta &= \sin\beta\sin(\varepsilon_A + \alpha) \\
\sin\eta\sin\delta &= \cos(\varepsilon_A + \alpha) \\
\sin\eta\cos\delta &= -\cos\beta\sin(\varepsilon_A + \alpha)
\end{aligned} \tag{2}$$

and, from triangle $\Upsilon_\circ P_t P_\circ$, we got the precession angles θ_A, ζ_A:

$$\begin{aligned} \cos\theta_A &= -\sin\eta\sin(\mu+\delta-\varepsilon_\circ) \\ \sin\theta_A\sin\zeta_A &= -\sin\eta\cos(\mu+\delta-\varepsilon_\circ) \\ \sin\theta_A\cos\zeta_A &= \cos\eta. \end{aligned} \tag{3}$$

From the triangle $P_\circ P_t C_\circ$ followed the precession parameters ω_A, ψ_A:

$$\begin{aligned} \cos\omega_A &= \cos\varepsilon_\circ\cos\theta_A + \sin\varepsilon_\circ\sin\theta_A\sin\zeta_A \\ \sin\omega_A\sin\psi_A &= \sin\theta_A\cos\zeta_A \\ \sin\omega_A\cos\psi_A &= \sin\varepsilon_\circ\cos\theta_A - \cos\varepsilon_\circ\sin\theta_A\sin\zeta_A, \end{aligned} \tag{4}$$

and from the triangles $P_t C C_\circ$, $P_\circ P_t C_\circ$ the parameters χ_A, z_A:

$$\begin{aligned} \sin\varepsilon_A\sin\chi_A &= P_A\cos\psi_A + Q_A\sin\psi_A \\ \sin\varepsilon_A\cos\chi_A &= \cos\pi_A\sin\omega_A - (P_A\sin\psi_A - Q_A\cos\psi_A)\cos\omega_A \\ \sin\theta_A\sin(z_A+\chi_A) &= \sin\omega_A\cos\varepsilon_\circ - \cos\omega_A\sin\varepsilon_\circ\cos\psi_A \\ \sin\theta_A\cos(z_A+\chi_A) &= \sin\varepsilon_\circ\sin\psi_A. \end{aligned} \tag{5}$$

Instead of precession angles θ_A, z_A, ζ_A we use direction cosines $X_A = \sin\theta_A\cos\zeta_A$, $Y_A = -\sin\theta_A\sin\zeta_A$, $V_A = \sin\theta_A\sin z_A$, $W_A = \sin\theta_A\cos z_A$; the angles θ_A, ζ_A and z_A exhibit large discontinuities (of about 94° for θ_A, 180° for ζ_A and z_A) at irregular intervals: there is a change of sign approximately each 26,000 years. This makes the long-term analytical approximation of these precession angles extremely difficult, while the direction cosines are continuous.

The time series of all parameters calculated above were then approximated by a cubic polynomial plus up to 14 long-periodic terms of the general form (T is the time in centuries from J2000.0, P_i is the period and n the number of periodic terms)

$$a + bT + cT^2 + dT^3 + \sum_{i=1}^{n}\left(C_i\cos 2\pi T/P_i + S_i\sin 2\pi T/P_i\right), \tag{6}$$

so that the fit is best around J2000.0. This was assured by choosing appropriate weights (equal to 10^4 in the central part and to $1/T^2$ outside this interval). The periods were found beforehand using the Vaníček's method [9], modified by Vondrák [10], and verified with the ones found by Laskar et al. [5], [6] from much longer time series. Weighted least-squares estimation was then used to determine the sine/cosine amplitudes of individual periodic terms.

We derived the long-term expressions of the following precession parameters, some of them being precession angles, some direction cosines (expressed in terms of certain precession angles):

- precession angles: $p_A, \varepsilon_A, \omega_A, \psi_A, \chi_A, \varphi, \gamma, \psi$;
- direction cosines: $P_A = \sin\pi_A\sin\Pi_A$, $Q_A = \sin\pi_A\cos\Pi_A$, $X_A = \sin\theta_A\cos\zeta_A$, $Y_A = \sin\theta_A\sin\zeta_A$, $V_A = \sin\theta_A\sin z_A$, $W_A = \sin\theta_A\cos z_A$.

We also derived the expression for the CIO locator (the part that is due to precession), s_A. All these angles are depicted in Fig. 2.

2.1. Example

As a typical example, the long-term expressions of direction cosines of the pole P_t, X_A, Y_A (in arcseconds), are given below:

$$X_A = 5453.282155 + 0.4252841T - 0.00037173T^2 - 152 \times 10^{-9}T^3 + \sum_X,$$
$$Y_A = -73750.930350 - 0.7675452T - 0.00018725T^2 + 231 \times 10^{-9}T^3 + \sum_Y,$$

where the cosine/sine amplitudes of the periodic parts $\sum_X$, $\sum_Y$ are displayed in Tab. 1. The comparisons of the long-term models of precession angles X_A(top) and Y_A (bottom) are shown in Fig. 3. The model and integrated values are so close that they are graphically indistinguishable. One can readily see that the expressions for X_A, Y_A of IAU2006 model quickly deviate from the former ones. The behavior of other precession parameters is similar.

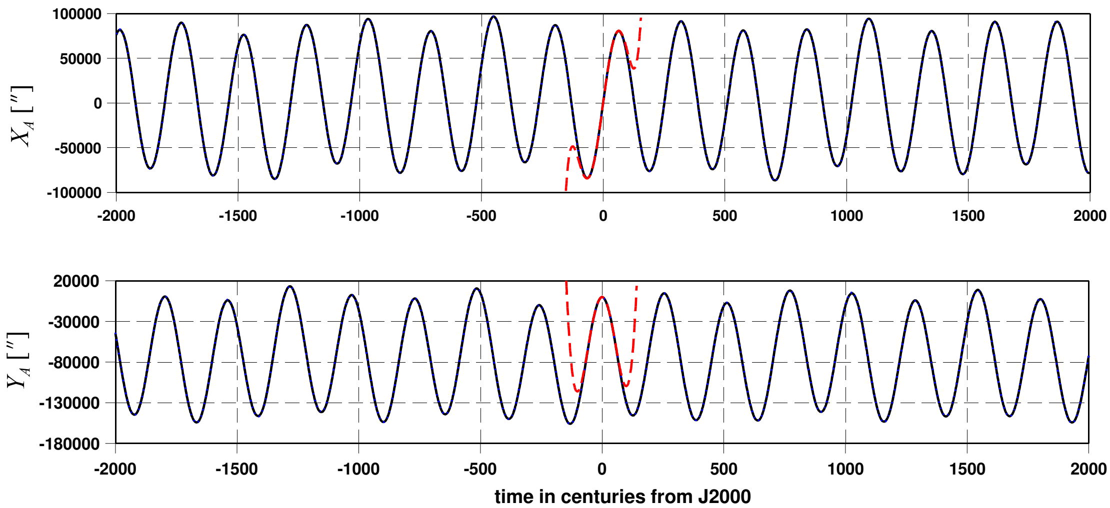

Figure 3: Long-term model of precession parameters X_A, Y_A – new model (dotted), integrated values (solid), and IAU2006 (dashed).

2.2. Alternative parametrization of precession matrix

Different combinations of the precession angles derived above can be used to compute precession matrix $\mathbf{P}$, necessary to transform coordinates of celestial bodies from the fundamental epoch J2000.0 to any epoch T:

- 'Lieske' parametrization [7]: $\mathbf{P} = \mathbf{R}_3(-z_A).\mathbf{R}_2(\theta_A).\mathbf{R}_3(-\zeta_A)$,
- 'Capitaine' parametrization [2]: $\mathbf{P} = \mathbf{R}_3(\chi_A).\mathbf{R}_1(-\omega_A).\mathbf{R}_3(-\psi_A).\mathbf{R}_1(\varepsilon_\circ)$,
- 'Williams-Fukushima' parametrization [4]: $\mathbf{P} = \mathbf{R}_1(-\varepsilon_A).\mathbf{R}_3(-\psi).\mathbf{R}_1(\varphi).\mathbf{R}_3(\gamma)$,

in which $\mathbf{R}_i(\alpha)$ denotes the rotation matrix around $i-$th axis by angle α. In the classical 'Lieske' parametrization the precession angles z_A, θ_A, ζ_A can be easily expressed in terms of direction cosines X_A, Y_A, V_A, W_A. Quite naturally, all these methods should theoretically lead to the same result.

term	C/S	$X_A['']$	$Y_A['']$	P[cy]
p	C_1	-819.940624	75004.344875	256.75
	S_1	81491.287984	1558.515853	
$-\sigma_3$	C_2	-8444.676815	624.033993	708.15
	S_2	787.163481	7774.939698	
$p - g_2 + g_5$	C_3	2600.009459	1251.136893	274.20
	S_3	1251.296102	-2219.534038	
$p + g_2 - g_5$	C_4	2755.175630	-1102.212834	241.45
	S_4	-1257.950837	-2523.969396	
$-s_1$	C_5	-167.659835	-2660.664980	2309.00
	S_5	-2966.799730	247.850422	
$-s_6$	C_6	871.855056	699.291817	492.20
	S_6	639.744522	-846.485643	
$p + s_4$	C_7	44.769698	153.167220	396.10
	S_7	131.600209	-1393.124055	
$p + s_1$	C_8	-512.313065	-950.865637	288.90
	S_8	-445.040117	368.526116	
$p - s_1$	C_9	-819.415595	499.754645	231.10
	S_9	584.522874	749.045012	
	C_{10}	-538.071099	-145.188210	1610.00
	S_{10}	-89.756563	444.704518	
	C_{11}	-189.793622	558.116553	620.00
	S_{11}	524.429630	235.934465	
$2p + s_3$	C_{12}	-402.922932	-23.923029	157.87
	S_{12}	-13.549067	374.049623	
	C_{13}	179.516345	-165.405086	220.30
	S_{13}	-210.157124	-171.330180	
	C_{14}	-9.814756	9.344131	1200.00
	S_{14}	-44.919798	-22.899655	

Table 1: Periodic terms in X_A, Y_A.

3. Estimation of model accuracy, comparison with other models

In paper [13] the accuracy was estimated using a simple expression based on the average uncertainty of all parameters (derived from the fit to integrated values) and weights at different epochs. Here a rigorous formula is used, based on the full variance-covariance matrix. The result is depicted in Fig. 4, where the accuracy of each estimated parameter is given and compared with the one from the paper [13]. It is clear that our previous estimate was too pessimistic – the rigorous estimate yields much smaller uncertainties for all parameters, in some cases as much as two orders of magnitude lower.

The comparison of the new long-term solution with other models of precession

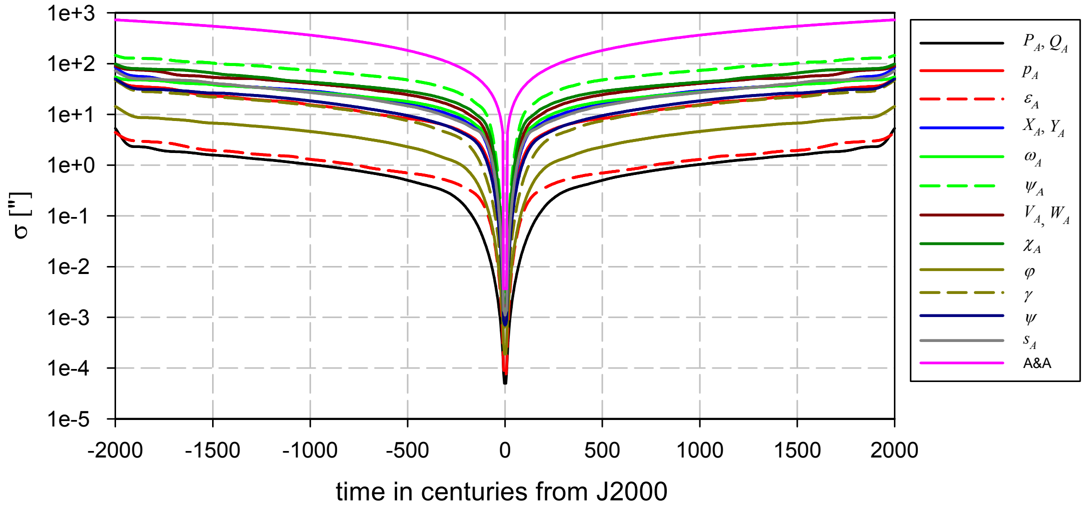

Figure 4: Estimated accuracy of all precession parameters.

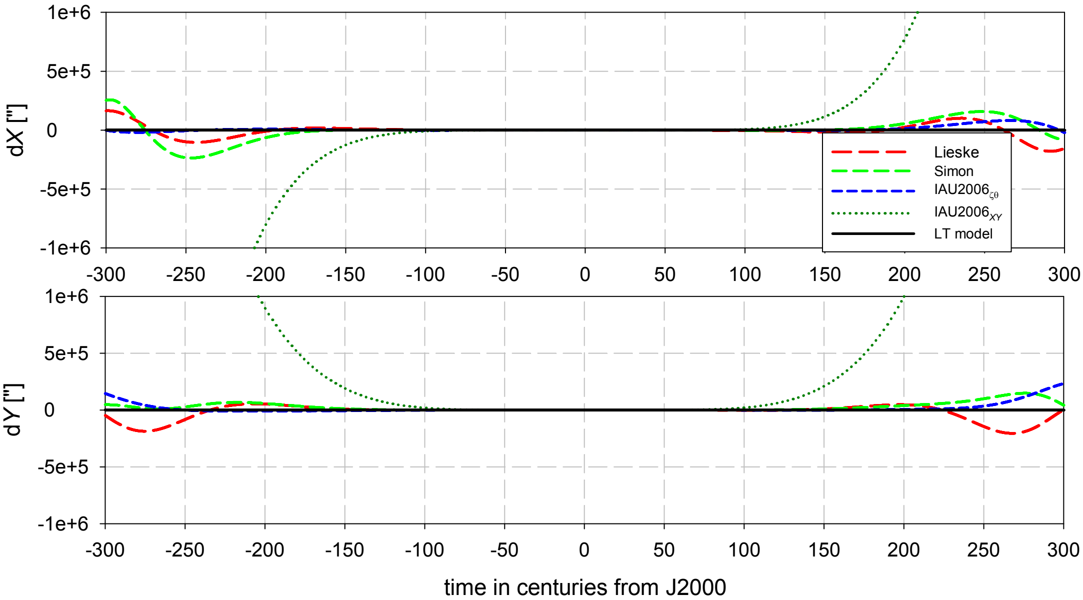

Figure 5: Comparison of different precession models with integrated values.

(X_A and Y_A parameters only) is given in Figs 5 and 6. X_A and Y_A values as computed from the values of ζ_A, θ_A by Lieske et al. [7], Simon et al. [8] and Capitaine et al. [2] (denoted as Lieske, Simon, IAU2006$_{\zeta\theta}$) and computed directly from the X_A, Y_A expressions of Capitaine et al. [2] and paper [13] (denoted as IAU2006$_{XY}$, LT model) are compared with the numerically integrated values.

Fig. 5 depicts the comparison in the interval ±300 centuries from J2000.0, while Fig. 6 shows only the central part (±10 centuries from J2000.0) at an enlarged scale.

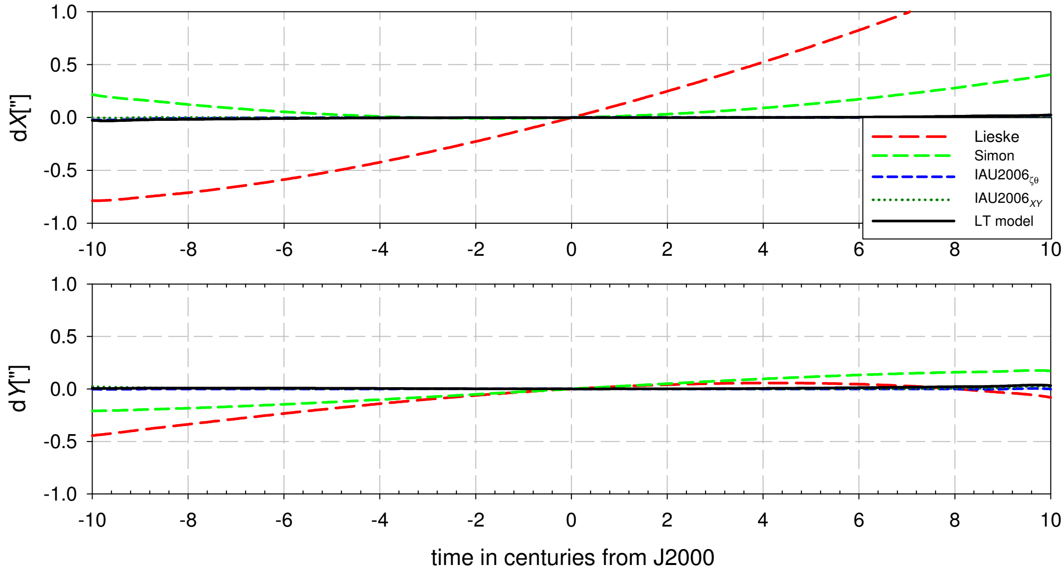

Figure 6: Comparison of precession models – closeup of the central part.

One can see that the direct IAU2006 expressions for direction cosines X_A, Y_A yield much worse results than using the expressions for 'traditional' precession angles ζ_A, θ_A for more distant epochs. The new LT model is indistinguishable from the integration at this scale, whereas all other models display deviations reaching 50 degrees for epochs more distant than 200 centuries. Fig. 6 clearly demonstrates the correction of precession rate, and also the quadratic term in obliquity, introduced in all models with respect to Lieske et al. [7]. On the other hand, all models shown are consistent with the numerically integrated precession within one arcsecond or so in the interval ± 10 centuries from J2000.0.

4. Conclusions

The presently adopted IAU2006 model provides high accuracy over a few centuries around the epoch J2000.0. For longer periods, polynomial development of precession angles ζ_A, θ_A should be preferable to direct X_A, Y_A expressions. More than five thousand years from the fundamental epoch J2000.0 the model rapidly goes away from reality. The new model of precession, developed in paper [13] and valid over ± 200 millennia, is presented. Its accuracy is comparable to IAU2006 model in the interval of several centuries around J2000.0, and it fits the numerically integrated position of the pole for longer intervals, with gradually decreasing accuracy (several arcminutes ± 200 thousand years away from J2000.0). The estimated accuracy, as given in paper [13], is too pessimistic.

Acknowledgements

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