Motohiro Sobajima; Giorgio Metafune

An elementary proof of asymptotic behavior of solutions of U” = VU

In: Karol Mikula (ed.): Proceedings of Equadiff 14, Conference on Differential Equations and Their Applications, Bratislava, July 24-28, 2017. Slovak University of Technology in Bratislava, SPEKTRUM STU Publishing, Bratislava, 2017. pp. 369–376.

Persistent URL: http://dml.cz/dmlcz/703035

Proceedings of EQUADIFF 2017
pp. 369–376

AN ELEMENTARY PROOF OF ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF $U'' = VU$ *

MOTOHIRO SOBAJIMA† AND GIORGIO METAFUNE‡

Abstract. We provide an elementary proof of the asymptotic behavior of solutions of second order differential equations without successive approximation argument.

Key words. Elementary proof, second-order ordinary differential equations, asymptotic behavior.

AMS subject classifications. 34E10

1. Introduction. The asymptotic behavior of the solutions of the ordinary differential equation

$$u''(x) = V(x)u(x), \qquad x \in (0,\infty) \tag{1.1}$$

is an important tool in various fields of mathematics and mathematical physics, in particular when special functions are involved. It can be found in [3, Section 6.2] and partially in [1, Chapter 10] and in [2, Chapter IV] that if $V(x) = f(x) + g(x)$, that is,

$$u''(x) = \big(f(x) + g(x)\big)u(x), \qquad x \in (0,\infty) \tag{1.2}$$

and

$$\psi_{f,g} := |f|^{-\frac{1}{4}} \left(-\frac{d^2}{dx^2} + g\right) |f|^{-\frac{1}{4}} \text{ is absolutely integrable in } (0,\infty), \tag{1.3}$$

then two solutions of (1.2) behave like

$$u(x) \approx |f|^{-1/4} e^{\pm \int_0^x |f(s)|^{1/2}\, ds}, \quad u(x) \approx |f|^{-1/4} e^{\pm i \int_0^x |f(s)|^{1/2}\, ds}.$$

The proof is usually done treating first the cases $f = \pm 1$ and then reducing to them the general case, by the Liouville transformation. We follow the same approach but simplify the cases $f = \pm 1$ by using Gronwall's Lemma, instead of successive approximations. In order to keep the exposition at an elementary level, we avoid also Lebesgue integration and dominated convergence (which could shorten some proofs); note that we only use the notation $f \in L^1(I)$ when f is absolutely integrable in I. We consider both the behavior at infinity and near isolated singularities and apply the results to Bessel functions. We also recall that the general case

$$u''(x) + g(x)u'(x) = V(x)u(x)$$

can be reduced to the form (1.1) (with another V) by writing $u = \frac{1}{2}(\exp \int g)v$.

*This work is partially supported by Grant-in-Aid for Young Scientists Research (B) No.16K17619.

†Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda-shi, Chiba-ken 278-8510, Japan (msobajima1984@gmail.com).

‡Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, Via Per Arnesano, 73100, Lecce, Italy (giorgio.metafune@unisalento.it).

This kind of analysis can be applied to the spectral analysis for Schrödinger operator with singular potentials (for example $S = -\Delta + V(|x|)$ with $V(r) \sim r^{-\delta}$ near the origin). Actually, the essential selfadjointness of the Schrödinger operator S can be treated by using the limit-point and limit-circle criteria (see e.g., Reed–Simon [4]) which require the behavior of two solutions to $u - u'' + \frac{N-1}{r}u + Vu = 0$. The behavior of two solutions above leads also to resolvent estimates for S. From this view-piont, the elemental consideration in the present paper helps in understanding various spectral phenomena for second-order differential operators.

2. Behavior near infinity in the simplest cases. First we consider the cases $f \equiv 1$ and $f \equiv -1$ and we prove the following results to which the general case reduces.

PROPOSITION 2.1. *If $f = 1$, $g \in L^1(0,\infty)$, then there exist two solutions u_1 and u_2 of (1.2) such that, as $x \to \infty$,*

$$e^{-x}u_1(x) \to 1, \qquad e^{-x}u_1'(x) \to 1, \tag{2.1}$$

$$e^{x}u_2(x) \to 1, \qquad e^{x}u_2'(x) \to -1. \tag{2.2}$$

PROPOSITION 2.2. *If $f = -1$, $g \in L^1(0,\infty)$, then there exist two solutions v_1 and v_2 of (1.2) such that, as $x \to \infty$,*

$$e^{-ix}u_1(x) \to 1, \qquad e^{-ix}u_1'(x) \to i, \tag{2.3}$$

$$e^{ix}u_2(x) \to 1, \qquad e^{ix}u_2'(x) \to -i. \tag{2.4}$$

By variation of parameters, every solution of (1.2) can be written as

$$u(x) = c_1e^{\zeta x} + c_2e^{-\zeta x} + \frac{1}{2\zeta}\int_a^x (e^{\zeta(x-s)} - e^{-\zeta(x-s)})g(s)u(s)\,ds, \quad x \in [a,\infty), \tag{2.5}$$

with $c_1, c_2 \in \mathbb{C}$, $\zeta = 1, i, -i$ and $a > 0$. In the following Lemma we choose $c_1 = 1, c_2 = 0$ to construct a solution which behaves like $e^{\zeta x}$ as $x \to \infty$, $\zeta = 1, i, -i$.

LEMMA 2.3. *Let $\zeta \in \{1, i, -i\}$, $a > 0$ and $g \in L^1(a,\infty)$. If $u \in C^2([a,\infty))$ satisfies*

$$u(x) = e^{\zeta x} + \frac{1}{2\zeta}\int_a^x (e^{\zeta(x-s)} - e^{-\zeta(x-s)})g(s)u(s)\,ds, \qquad x \in [a,\infty),$$

then $z(x) := e^{-\zeta x}u(x)$ satisfies

$$|z(x)| \le e^{\int_a^x |g(r)|\,dr}, \qquad x \in [a,\infty) \tag{2.6}$$

$$\|zg\|_{L^1(a,\infty)} \le e^{\|g\|_{L^1(a,\infty)}} - 1. \tag{2.7}$$

Proof. Note that

$$z(x) = 1 + \frac{1}{2\zeta}\int_a^x (1 - e^{-2\zeta(x-s)})g(s)z(s)\,ds, \quad x \in [a,\infty).$$

Since $|1 - e^{-2\zeta(x-s)}| \le 2$ for $s \le x$, we see that for $x \ge a$,

$$|z(x)| \le 1 + \left|\frac{1}{2\zeta}\int_a^x (1 - e^{-2\zeta(x-s)})g(s)z(s)\,ds\right| \le 1 + \int_a^x |g(s)|\,|z(s)|\,ds.$$

Thus Gronwall's lemma implies (2.6), in particular z is bounded on $[a,\infty)$ and then $zg \in L^1(a,\infty)$. Moreover we have

$$\|zg\|_{L^1(a,\infty)} \le \int_a^\infty |g(s)|\, e^{\int_a^s |g(r)|\,dr}\, ds = e^{\|g\|_{L^1(a,\infty)}} - 1.$$

□

Proof of Proposition 2.1. Let $a > 0$ such that $\|g\|_{L^1(a,\infty)} < \log 2$ and let u be in Lemma 2.3 with $\zeta = 1$. Then u is one solution of (1.2) with $f = 1$. Set $z(x) = e^{-x}u(x)$. Then noting that as $x \to \infty$,

$$\begin{aligned}\left|\int_a^x e^{-2(x-s)} g(s)z(s)\,ds\right| &\le \int_a^{\frac{a+x}{2}} e^{-2(x-s)}|g(s)z(s)|\,ds + \int_{\frac{a+x}{2}}^x |g(s)z(s)|\,ds\\ &\le e^{-x+a}\|gz\|_{L^1(a,\infty)} + \|gz\|_{L^1(\frac{a+x}{2},\infty)} \to 0,\end{aligned}$$

we see that z satisfies

$$\begin{aligned} z(x) &\to z_\infty := 1 + \int_a^\infty g(s)z(s)\,ds \quad \text{as } x \to \infty,\\ z'(x) &= \int_a^x e^{-2(x-s)} g(s)z(s)\,ds \to 0 \quad \text{as } x \to \infty.\end{aligned}$$

By (2.7), we deduce that $\|zg\|_{L^1(a,\infty)} < 1$. Therefore $|z_\infty - 1| \le \|zg\|_{L^1(a,\infty)} < 1$ and hence $z_\infty \ne 0$. The function $u_1(x) := z_\infty^{-1} e^x z(x)$ satisfies (2.1). Moreover, since u_1^{-2} is integrable near ∞, another solution of (1.2) is given by

$$u_2(x) = 2u_1(x) \int_x^\infty \frac{1}{u_1(s)^2}\,ds. \tag{2.8}$$

Integrating by parts we deduce that, as $x \to \infty$,

$$\begin{aligned} e^x u_2(x) &= 2z_\infty e^{2x} z(x) \int_x^\infty \frac{1}{e^{2s}[z(s)]^2}\,ds\\ &= z_\infty e^{2x} z(x) \left(-\left[\frac{1}{e^{2s}[z(s)]^2}\right]_{s=x}^{s=\infty} - 2\int_x^\infty \frac{z'(s)}{e^{2s}[z(s)]^3}\,ds \right) \to 1\end{aligned}$$

and

$$[e^x u_2(x)]' = 2z_\infty e^{2x} z'(x) \int_x^\infty \frac{1}{e^{2s}[z(s)]^2}\,ds + 2e^x u_2(x) - \frac{2z_\infty}{z(x)} \to 0.$$

□

Proof of Proposition 2.2. Let $a > 0$ such that $\|g\|_{L^1(a,\infty)} < \log 2$ and let $\tilde u_1$ and $\tilde u_2$ be as in Lemma 2.3 with $\zeta = i$ and with $\zeta = -i$, respectively. Noting that both $\tilde u_1$ and $\tilde u_2$ satisfy (1.2) with $f = -1$, and setting $z_1(x) = e^{-ix}\tilde u_1(x)$ and $z_2(x) = e^{ix}\tilde u_2(x)$, we have as $x \to \infty$

$$\begin{aligned} e^{2ix}\left(z_1(x) - 1 - \frac{1}{2i}\int_a^\infty g(s)z_1(s)\,ds\right) &\to \frac{1}{2i}\int_a^\infty e^{2is} g(s)z_1(s)\,ds,\\ e^{-2ix}\left(z_2(x) - 1 + \frac{1}{2i}\int_a^\infty g(s)z_2(s)\,ds\right) &\to -\frac{1}{2i}\int_a^\infty e^{-2is} g(s)z_2(s)\,ds\end{aligned}$$

and

$$e^{2ix}z_1'(x) \to \int_a^\infty e^{2is}g(s)z_1(s)\,ds, \qquad e^{-2ix}z_2'(x) \to \int_a^\infty e^{-2is}g(s)z_2(s)\,ds.$$

It follows that $\tilde{u}_1 \approx \xi_1 e^{ix} + \xi_2 e^{-ix}$, $\tilde{u}_1' \approx i\xi_1 e^{ix} - i\xi_2 e^{-ix}$ and $\tilde{u}_2 \approx \eta_1 e^{ix} + \eta_2 e^{-ix}$, $\tilde{u}_2' \approx i\eta_1 e^{ix} - i\eta_2 e^{-ix}$ as $x \to \infty$ where

$$\xi_1 = 1 + \frac{1}{2i}\int_a^\infty g(s)z_1(s)\,ds, \qquad \xi_2 = -\frac{1}{2i}\int_a^\infty e^{2is}g(s)z_1(s)\,ds,$$

and similarly for η_1, η_2. From (2.7) we see that $|\xi_1| > 1/2$, $|\xi_2| < 1/2$, $|\eta_1| < 1/2$ and $|\eta_2| > 1/2$ and hence $|\xi_1\eta_2 - \xi_2\eta_1| > 0$ and $\tilde{u}_1$ and $\tilde{u}_2$ are linearly independent. Therefore we can construct solutions u_1 and u_2 which satisfy (2.3) and (2.4), respectively. □

We consider now the case $f = 0$, assuming extra conditions on g.

PROPOSITION 2.4. *Assume that $xg \in L^1(0,\infty)$. Then there exist two solutions u_1 and u_2 of*

$$u''(x) = g(x)u(x) \tag{2.9}$$

such that

$$x^{-1}u_1(x) \to 1, \qquad u_1'(x) \to 1,$$
$$u_2(x) \to 1, \qquad xu_2'(x) \to 0$$

as $x \to \infty$, respectively.

Proof. Set $u(x) := xz(x)$. Then $z'' + (2/x)z' = gz$ and, assuming $z'(a) = 0$ we obtain

$$z'(x) = x^{-2}\int_a^x s^2 g(s)z(s)\,ds. \tag{2.10}$$

Then assuming $z(a) = 1$

$$\begin{aligned} |z(x) - 1| &\le \int_b^x t^{-2}\left(\int_a^t s^2|g(s)z(s)|\,ds\right)dt \\ &= \int_a^x \left(\int_s^x t^{-2}\,dt\right)s^2|g(s)z(s)|\,ds \le \int_a^x s|g(s)z(s)|\,ds. \end{aligned} \tag{2.11}$$

Gronwall's lemma yields

$$|z(x)| \le e^{\int_a^x s|g(s)|\,ds}$$

hence z is bounded and $z' \in L^1(a,\infty)$ by (2.10). As in the proof of Proposition 2.1, $z(x) \to z_\infty \neq 0$ if a is sufficiently large. Moreover, since as $x \to \infty$,

$$|xz'(x)| \le \sqrt{\frac{a}{x}}\int_a^{\sqrt{ax}} s|g(s)z(s)|\,ds + \int_{\sqrt{ax}}^x s|g(s)z(s)|\,ds \to 0,$$

$u_1(x) := z_\infty^{-1}xz(x)$ satisfies the statement. Another solution u_2 of (1.2) is given by

$$u_2(x) := u_1(x)\int_x^\infty \frac{1}{u_1(s)^2}\,ds.$$

As in the proof of Proposition 3.1 we can verify that u_2 satisfies $u_2(x) \to 1$ and $xu_2'(x) \to 0$ as $x \to \infty$. □

Observe the integrability condition for xg near ∞ is necessary. In fact, if $g(x) = cx^{-2}$ the above equation has solutions x^α if $\alpha^2 - \alpha = c$.

3. Behavior near infinity in the general case. We recall that the function $\psi_{f,g}$ is defined in (1.3) and set $v_j(x) = |f|^{1/4}u_j(x)$, $j = 1, 2$ if u_1, u_2 are solutions of (1.2). The hypothesis $|f|^{1/2}$ not summable near ∞ guarantees that the Liouville transformation Φ of Lemma 3.3 maps (a,∞) onto $(0,\infty)$, so that the results of the previous section apply. When it is not satisfied Φ maps (a,∞) onto a bounded interval $(0,b)$ and the behavior of the solutions of (3.5) near b is more elementary (in some cases one can use Proposition 2.4).

Proposition 3.1. *Assume that $f(x) > 0$ in (a,∞), $|f|^{1/2} \notin L^1(a,\infty)$ and $\psi_{f,g} \in L^1(a,\infty)$. Then there exist two solutions u_1 and u_2 of* (1.2) *such that as $x \to \infty$*

$$e^{-\int_a^x |f(r)|^{1/2}dr}v_1(x) \to 1, \qquad |f(x)|^{-1/2}e^{-\int_a^x |f(r)|^{1/2}dr}v_1'(x) \to 1, \tag{3.1}$$

$$e^{\int_a^x |f(r)|^{1/2}dr}v_2(x) \to 1, \qquad |f(x)|^{-1/2}e^{\int_a^x |f(r)|^{1/2}dr}v_2'(x) \to -1. \tag{3.2}$$

Proposition 3.2. *Assume that $f(x) < 0$ in (a,∞), $|f|^{1/2} \notin L^1(a,\infty)$ and $\psi_{f,g} \in L^1(a,\infty)$. Then there exists two solutions u_1 and u_2 of* (1.2) *such that as$x \to \infty$*

$$e^{-i\int_a^x |f(r)|^{1/2}dr}v_1(x) \to 1, \qquad |f(x)|^{-1/2}e^{-i\int_a^x |f(r)|^{1/2}dr}v_1'(x) \to i, \tag{3.3}$$

$$e^{i\int_a^x |f(r)|^{1/2}dr}v_2(x) \to 1, \qquad |f(x)|^{-1/2}e^{i\int_a^x |f(r)|^{1/2}dr}v_2'(x) \to -i. \tag{3.4}$$

The proof is based on the well-known Liouville transformation that we recall below.

Lemma 3.3. *Let $a > 0$ and assume that $f \in C^2([a,\infty))$ satisfies $|f(x)| > 0$, $|f|^{1/2} \notin L^1(a,\infty)$. Define $\Phi \in C^2([a,\infty))$ by*

$$\Phi(x) := \int_a^x |f(r)|^{1/2}\,dr, \quad x \in [a,\infty).$$

Then $\Phi^{-1} : [0,\infty) \to [a,\infty)$ and if u satisfies (1.2) *the function*

$$w(y) := |f(\Phi^{-1}(y))|^{1/4}u(\Phi^{-1}(y)), \quad y \in [0,\infty)$$

satisfies

$$w''(y) = \left(\frac{f(\Phi^{-1}(y))}{|f(\Phi^{-1}(y))|} + \frac{\psi_{f,g}(\Phi^{-1}(y))}{|f(\Phi^{-1}(y))|^{1/2}}\right)w(y). \tag{3.5}$$

Proof. Note that $\Phi'(x) = |f(x)|^{1/2}$ and $\frac{d(\Phi^{-1})}{dy}(y) = |f(\Phi^{-1}(y))|^{-1/2}$. Setting

$w(y) = |f(\Phi^{-1}(y))|^{1/4}u(\Phi^{-1}(y))$ (and using $\xi = \Phi^{-1}(y)$ for simplicity), we have

$$\begin{aligned} w'(y) &= \frac{d}{dx}\left[|f|^{1/4}u\right](\xi)\frac{d(\Phi^{-1})}{dy}(y) \\ &= |f(\xi)|^{-1/4}u'(\xi) + \left[|f|^{-1/2}\frac{d}{dx}|f|^{1/4}\right](\xi)u(\xi) \\ &= \left[|f|^{-1/4}u' - \frac{d}{dx}(|f|^{-1/4})u\right](\xi), \\ w''(y) &= \frac{d}{dx}\left[|f|^{-1/4}u' - \frac{d}{dx}(|f|^{-1/4})u\right](\xi)\frac{d(\Phi^{-1})}{dy}(y) \\ &= |f(\xi)|^{-3/4}u''(\xi) - \left[|f|^{-1/2}\frac{d^2}{dx^2}|f|^{-1/4}\right](\xi)u(\xi) \\ &= |f(\xi)|^{-1}(f(\xi)+g(\xi))w(y) - \left[|f|^{-3/4}\frac{d^2}{dx^2}|f|^{-1/4}\right](\xi)w(y). \end{aligned}$$

Thus we obtain (3.5). □

Proof. [Proof of Propositions 3.1 and 3.2] It suffices to apply Propositions 2.1 and 2.2 to the respective cases $f > 0$ and $f < 0$. Set $h(y) = \psi_{f,g}(\Phi^{-1}(y))|f(\Phi^{-1}(y))|^{-1/2}$. Then

$$\int_0^b |h(y)|\,dy = \int_a^\infty |\psi_{f,g}(x)|\,dx.$$

Therefore Propositions 2.1 and 2.2 are applicable to $w'' = \pm w + hw$, respectively. Finally, using Lemma 3.3 and taking $u(x) = |f(x)|^{-1/4}w(\Phi(x))$, we obtain the respective assertions in Propositions 3.1 and 3.2. □

4. Behavior near interior singularities. If f and g have local singularities at x_0, then the behavior of solutions near x_0 is also considerable. For simplicity, we take $x_0 = 0$. The following propositions are meaningful when $|f|^{1/2}$ is not integrable near 0, in particular when $|f|^{1/2} = cx^{-1}$. We recall that $v_j(x) = |f(x)|^{1/4}u_j(x)$, $j = 1, 2$.

PROPOSITION 4.1. *Assume that $f(x) > 0$ in $(0,\infty)$ and $\psi_{f,g} \in L^1(0,\infty)$. Then there exist two solutions u_1 and u_2 of (1.2) such that as $x \downarrow 0$*

$$\begin{aligned} e^{-\int_x^1 |f(r)|^{1/2}dr}v_1(x) \to 1, &\qquad |f(x)|^{-1/2}e^{-\int_x^1 |f(r)|^{1/2}dr}v_1'(x) \to -1, \\ e^{\int_x^1 |f(r)|^{1/2}dr}v_2(x) \to 1, &\qquad |f(x)|^{-1/2}e^{\int_x^1 |f(r)|^{1/2}dr}v_2'(x) \to 1. \end{aligned}$$

PROPOSITION 4.2. *Assume that $f(x) < 0$ in $(0,\infty)$ and $\psi_{f,g} \in L^1(0,\infty)$. Then there exist two solutions u_1 and u_2 of (1.2) such that as $x \downarrow 0$*

$$\begin{aligned} e^{-\int_x^1 |f(r)|^{1/2}dr}v_1(x) \to 1, &\qquad |f(x)|^{-1/2}e^{-\int_x^1 |f(r)|^{1/2}dr}v_1'(x) \to -i, \\ e^{\int_x^1 |f(r)|^{1/2}dr}v_2(x) \to 1, &\qquad |f(x)|^{-1/2}e^{\int_x^1 |f(r)|^{1/2}dr}v_2'(x) \to i. \end{aligned}$$

Proof of Propositions 4.1 and 4.2. Setting $w(s) := su(s^{-1})$ we see that

$$\begin{aligned} w''(s) &= s^{-3}u''(s^{-1}) \\ &= s^{-3}(f(s^{-1})+g(s^{-1}))u(s^{-1}) = s^{-4}(f(s^{-1})+g(s^{-1}))w(s). \end{aligned}$$

Let $\tilde{f}(s) := s^{-4}f(s^{-1})$ and $\tilde{g}(s) := s^{-4}g(s^{-1})$. Noting that

$$\begin{aligned}\psi_{\tilde{f},\tilde{g}}(s) &= s|f(s^{-1})|^{-1/4}\left(-\frac{d^2}{ds^2}+s^{-4}g(s^{-1})\right)\left(s|f(s^{-1})|^{-1/4}\right)\\ &= s^{-2}|f(s^{-1})|^{-1/4}\left(-\frac{d^2}{dx^2}|f|^{-1/4}+g|f|^{-1/4}\right)(s^{-1})\\ &= s^{-2}\psi_{f,g}(s^{-1}),\end{aligned}$$

we have $\psi_{\tilde{f},\tilde{g}} \in L^1((0,\infty))$, and hence Propositions 3.1 and 3.2 can be applied. Since

$$\int_1^s |\tilde{f}(r)|^{1/2}dr = \int_{1/s}^1 |f(t)|^{1/2}dt,$$

we obtain the respective assertions in Propositions 4.1 and 4.2. □

5. Examples from special functions. Some examples illustrate the application of the results of the previous sections.

Example 1 (Modified Bessel functions). *We consider the modified Bessel equation of order* ν

$$u'' + \frac{u'}{r} - \left(1+\frac{\nu^2}{r^2}\right)u = 0, \tag{5.1}$$

All solutions of (5.1) *can be written through the modified Bessel functions* I_ν *and* K_ν. *Both* I_ν *and* K_ν *are positive,* I_ν *is monotone increasing and* K_ν *is monotone decreasing (see e.g., [3, Theorem 7.8.1]). Proposition 2.1 and Proposition 4.1 give the precise behavior of* I_ν *and* K_ν *near* ∞ *and near* 0, *respectively. In fact,* (5.1) *can be written as*

$$(\sqrt{r}u)'' = \left(1+\frac{4\nu^2-1}{4r^2}\right)(\sqrt{r}u). \tag{5.2}$$

Since $1/r^2$ *is integrable near* ∞, *choosing* $f = 1$ *and* $g = \frac{4\nu^2-1}{4r^2}$, *we see from Proposition 2.1 that*

$$\sqrt{r}e^{-r}I_\nu(r) \to c_1 \neq 0 \quad \textit{and} \quad \sqrt{r}e^{r}K_\nu(r) \to c_2 \neq 0 \qquad \textit{as } r \to \infty.$$

Moreover, if $\nu \neq 0$, *then choosing* $f(r) = \frac{\nu^2}{r^2}$ *and* $g(r) = 1-\frac{1}{4r^2}$, *that is,* $\psi_{f,g}(r) = r/\nu$, *from Proposition 4.1 we have*

$$r^{-\nu}I_\nu(r) \to c_3 \neq 0 \quad \text{and} \quad r^{\nu}K_\nu(r) \to c_4 \neq 0 \qquad \textit{as } r \downarrow 0.$$

If $\nu = 0$, *then putting* $w(s) = u(e^{-s})$ *we obtain*

$$w''(s) = e^{-2s}w(s), \qquad s \in \mathbb{R}.$$

Therefore using Proposition 2.4 with $\tilde{g}(s) = e^{-2s}$ *and taking* $u(x) = w(-\log x)$, *we have*

$$I_0(r) \to c_5 \neq 0 \quad \textit{and} \quad |\log r|^{-1}K_0(r) \to c_6 \neq 0 \qquad \textit{as } r \downarrow 0.$$

Example 2 (Fundamental solution of $\lambda-\Delta$). *For $n\ge 3$, $\lambda\ge 0$ the fundamental solution v_λ of $\lambda-\Delta$ can be computed by integrating the heat kernel:*

$$v_\lambda(r)=\int_0^\infty \frac{1}{(4\pi t)^{n/2}}e^{-\lambda t-\frac{r^2}{4t}}\,dt,$$

where $r=|x|$. Clearly $v_\lambda(r)\le v_0(r)=cr^{2-n}$, $v_\lambda(r)\to 0$ as $r\to\infty$. The function $v=v_\lambda$ satisfies

$$v''+\frac{n-1}{r}v'=\lambda v$$

or, setting $v=r^{(1-n)/2}w$,

$$w''=\left(\lambda+\frac{n^2-1}{4r^2}\right)w.$$

Proceeding as in the example above we see that $r^{2-n}v(r)\to c_1\ne 0$ as $r\to 0$ and $r^{(n-1)/2}e^{\sqrt{\lambda}r}v(r)\to c_2\ne 0$ as $r\to\infty$.

Example 3 (Bessel functions). *Next we consider the Bessel equation of order ν*

$$u''+\frac{u'}{r}+\left(1-\frac{\nu^2}{r^2}\right)u=0, \tag{5.3}$$

or equivalently,

$$(\sqrt{r}u)''=\left(-1+\frac{4\nu^2-1}{4r^2}\right)(\sqrt{r}u).$$

All solutions of (5.3) can be written through the Bessel functions J_ν and Y_ν. As in Example 1, from Propositions 4.1 (for $\nu>0$) and 2.4 (for $\nu=0$) we obtain the behavior of J_ν and Y_ν near 0

$$r^{-\nu}J_\nu(r)\to c_1\ne 0,\quad \text{and}\quad r^{\nu}Y_\nu(r)\to c_2\ne 0 \qquad \textit{as } r\downarrow 0$$

and if $\nu=0$,

$$|\log r|J_0(r)\to c_3\ne 0,\quad \text{and}\quad Y_0(r)\to c_4\ne 0 \qquad \textit{as } r\downarrow 0.$$

In view of Proposition 2.2 the behavior of J_ν and Y_μ near ∞ is given by

$$|\sqrt{r}J_\nu(r)-c_5\cos(r+\theta_1)|\to 0,\quad \text{and}\quad |\sqrt{r}Y_\nu(r)-c_6\cos(r+\theta_2)|\to 0,$$

as $r\to\infty$, where $c_5\ne 0$, $c_6\ne 0$ and $\theta_1,\theta_2\in[0,\pi)$ satisfy $\theta_1\ne\theta_2$.

REFERENCES

[1] R. Beals, R. Wong, "Special functions," A graduate text, Cambridge Studies in Advanced Mathematics **126**, Cambridge University Press, Cambridge, 2010.
[2] A. Erdélyi, "Asymptotic expansions," Dover Publications, Inc., New York, 1956.
[3] F.W.J. Olver, "Asymptotics and special functions," Computer Science and Applied Mathematics, Academic Press, New York-London, 1974.
[4] M. Reed, B. Simon, "Methods of modern mathematical physics. II. Fourier analysis, self-adjointness," Academic Press, New York-London, 1975.